ME 4024: Machine Dynamics

Course Notebook

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Contents

I Lecture 21

1 Dynamics Problem Solving 1-1

1.1 Dynamics Vocabulary . . . 1-1
1.2 Problem Solving . . . 1-2
1.3 Vector Operations . . . 1-3
1.3.1 Cartesian Coordinates . . . 1-3
1.3.2 Dot Product . . . 1-3
1.3.3 Cross Product . . . 1-4
1.3.4 Product Rule . . . 1-4
1.3.5 Partial and Total Derivatives . . . 1-4
1.3.6 Time Derivative of a Vector . . . 1-5
1.4 Kinematic Relationships for Rectilinear Motion . . . 1-6
1.4.1 Velocity, distance, and time . . . 1-6
1.4.2 Acceleration, velocity and time . . . 1-6
1.4.3 Displacement, velocity and acceleration . . . 1-6
1.4.4 Special Cases for Constant Acceleration . . . 1-6
1.5 Newton's Laws of Motion . . . 1-6
1.6 Inertial Reference Frames . . . 1-7
1.7 Force, Mass and Weight . . . 1-8
1.8 Example . . . 1-9
1.9 Tangent and Normal Coordinates . . . 1-11
1.10 Radial and Transverse Coordinates . . . 1-12
1.11 Homework Problem Set 1 . . . 1-13
1.12 Momentum . . . 1-16
1.12.1 Concept of Impulse . . . 1-18

1.12.2 System Momenta 1-19
1.12.3 Moment of Inertia 1-19
1.13 Collisions in 2-D 1-23
1.13.1 Center of Percussion 1-23
1.13.2 Collision Reconstruction Example Using The Conservation of Linear Momentum 1-24
1.13.3 Collision Reconstruction Example Using Angular Momentum 1-29
1.14 Homework Problem Set 2 1-34
1.15 Work and Energy 1-36
1.15.1 Kinetic Energy 1-36
1.15.2 Potential Energy 1-37
1.15.3 Work 1-37
1.15.3.1 Conservative Forces 1-37
1.15.3.2 Non-Conservative Forces 1-37
1.15.3.3 Work-less Forces 1-37
1.15.4 The Work-Energy Theorem 1-38
1.15.5 The Conservation of Mechanical Energy 1-38
1.16 Homework Problem Set 3 1-41
1.17 Principle of Virtual Work 1-43
1.18 Lagrange Equations 1-44
1.19 Simulation 1-45
1.19.1 State-Space Form 1-46
1.19.2 Euler's Method 1-47
1.19.3 Runge-Kutta 1-48
1.20 Homework Problem Set 4 1-49

2 Vibration 2-1

2.1 Free Vibration 2-1
2.1.1 Undamped Systems 2-1
2.1.2 Systems with Damping 2-2
2.2 Homework Problem Set 5 2-3
2.3 Forced Vibration 2-6
2.3.1 Harmonic Forcing 2-6
2.3.2 Forcing by Rotating Unbalance 2-7
2.3.3 Base Excitation 2-8

2.3.4 Frequency Response Functions 2-9
2.4 Homework Problem Set 6 2-10
2.5 Critical Shaft Speed 2-11
2.5.1 Single Degree of Freedom Model 2-11
2.5.2 Rayleigh's Method 2-12
2.6 Homework Problem Set 7 2-13

3 Dynamic Force Analysis 3-1
3.1 D'Alembert's Principle 3-1
3.2.1 Example 3-2
3.3 Dynamics of a 4-Bar Mechanism 3-7
3.3.1 Velocity Analysis (ME3212) 3-8
3.3.2 Free Body Diagrams 3-9
3.3.3 Equations of Motion 3-11
3.3.4 Solution Technique 3-12
3.4 Homework Problem Set 8 3-15
3.5 Dynamics of an Inverted Slider-Crank Mechanism 3-17
3.5.1 Kinematic Analysis 3-17

4 Balancing 4-1
4.1 Balancing Rotating Masses 4-1
4.1.1 Graphical Solution 4-2
4.1.2 Analytical Solution 4-3
4.1.3 Static Balance 4-4
4.2 Dynamic Balancing 4-7
4.3 Homework Problem Set 9 4-8
4.4 Field Balancing 4-9
4.5 Balancing Reciprocating Masses 4-10
4.5.1 Single Cylinder Engines 4-10
4.5.2 Multi Cylinder In-line Engines 4-11
4.6 Homework Problem Set 10 4-12

II Reference Material 4-15

5 Standards for Measurement 5-1

5.1 Measurement 5-1
5.2 Physical Quantities and Units of Measure 5-1
5.2.1 Fundamental Units 5-3
5.2.1.1 Standard of Length 5-3
5.2.1.2 Standard of Mass 5-4
5.2.1.3 Standard of Time 5-5
5.2.1.4 Standard of Electric Current 5-7
5.2.1.5 Standard of Temperature 5-7
5.2.1.6 Standard of Amount of Substance 5-8
5.2.1.7 Standard of Intensity of Light 5-8
5.2.2 Derived Units 5-8
5.2.2.1 Velocity and Speed 5-8
5.2.2.2 Acceleration 5-9
5.2.2.3 Force 5-9
5.2.2.4 Angles 5-11
5.2.2.5 Pressure 5-12
5.2.2.6 Density 5-13
5.2.2.7 Work and Energy 5-13
5.2.2.8 Torque and Moments 5-14
5.2.2.9 Power 5-15
5.2.3 Factors and Ratios 5-16
5.2.4 Dimensional Analysis 5-16
5.3 Accuracy and Precision 5-18
5.4 Numerical Precision 5-20
5.4.1 Significant Figures 5-21
5.4.2 Rounding 5-22
5.4.3 Truncating 5-23
5.5 Quantifying Uncertainty 5-24

6 Vectors 6-1

6.1 Vector and Scalar Quantities 6-1
6.2 Vector Basics and Drawing Vectors 6-3

6.3 Length of a Vector 6-5
6.4 Vector Addition 6-7
6.4.1 Vector Addition Example 6-8
6.5 Vector Subtraction 6-10
6.6 Vector Multiplication 6-10
6.6.1 Vector–Scalar Multiplication 6-11
6.6.2 Dot Product 6-11
6.6.3 Cross Product 6-13
6.7 Converting Polar Coordinates to Rectangular Coordinates 6-16
6.8 Resolving a Vector into Components 6-17
6.8.1 Example 6-18

7 Motion In One Dimension 7-1
7.1 The Study of Motion: Kinematics 7-1
7.2 Rectilinear Motion 7-3
7.2.1 Constant Velocity 7-4
7.2.2 Constant Acceleration 7-7
7.3 Uniformly Accelerated Motion 7-9
7.3.1 Relationship Between Acceleration and Velocity 7-9
7.3.2 Relationship between Acceleration, Time, and Displacement 7-10
7.3.3 Relationship between Acceleration, Velocity, and Displacement 7-12
7.4 Variable Acceleration 7-15
7.4.1 Kinematic Relations Using Calculus Concepts 7-16
7.4.2 Dealing with Variable Acceleration without Calculus 7-24
7.5 Relative Motion 7-28
7.5.1 Example 7-29

8 Motion in More than One Dimension 8-1
8.1 Degrees of Freedom 8-1
8.2 Motion in Space 8-1
8.3 Displacement 8-3
8.4 Speed and Velocity 8-4
8.4.1 Average Velocity 8-4
8.4.2 Instantaneous Velocity 8-6

8.5 Acceleration 8-7
8.5.1 Average Acceleration 8-7
8.5.2 Instantaneous Acceleration 8-8
8.6 Uniform Projectile Motion 8-8

9 Friction and Acceleration Factors 9-1
9.1 Coefficient of Friction 9-1
9.2 Acceleration (Drag) Factor 9-2
9.3 Determining Drag Factors 9-4
9.4 Effects of Uneven Braking on Drag Factor 9-8
9.4.1 Definition of Percentage of Braking 9-8
9.4.2 Application of Percentage of Braking 9-9
9.5 Gathering Road Friction Data 9-10
9.5.1 Tables 9-10
9.5.2 Drag Sleds 9-10
9.5.3 Tests with Skidding Vehicles 9-12
9.5.3.1 Measured Test Skids 9-12
9.5.3.2 Test Skids with Shot Markers (Bumper Guns) 9-14
9.5.3.3 Test Skids with Recording Radar 9-15
9.5.3.4 Test Skids with Accelerometers 9-16
9.6 Determining Drag Factors from Test Data 9-19
9.7 The Friction Circle and Lateral Friction 9-20

10 Dynamics and Newton's Laws of Motion 10-1
10.1 Newton's First Law 10-1
10.2 Newton's Second Law 10-2
10.3 The Concepts of Mass and Weight 10-3
10.4 Newton's Third Law 10-4
10.5 The Concept of Friction 10-4
10.6 Free Body Diagrams 10-6
10.6.1 Free Body Diagram Example 10-6
10.7 The Concept of Torque 10-7
10.7.1 Torque Example 10-9
10.8 The Concept of Equilibrium 10-10
10.8.1 Equillibrium Example 10-11

10.9 Center of Mass 10-16
10.9.1 Determining Center of Mass 10-17
10.9.2 Center of Mass Height 10-18
10.9.3 Center of Mass with Cargo 10-23
10.9.4 Center of Mass with Cargo Example 10-24
10.10 Dynamic Weight Shift 10-26

11 Linear Momentum 11-1
11.1 Linear Momentum and Impulse 11-1
11.2 Conservation of Linear Momentum 11-8
11.3 In-Line Momentum 11-10
11.4 Elastic and Inelastic Collisions 11-15
11.5 The Presence of External Forces 11-21

12 Collision Analysis Using Conservation of Linear Momentum 12-1
12.1 Introduction 12-1
12.2 Collision Types and Configurations 12-2
12.2.1 Collinear, Central Collisions 12-2
12.2.2 Collinear, Non-Central Collisions 12-2
12.2.3 Two-Dimensional, Central Collisions 12-3
12.2.4 Two-Dimensional, Non-Central Collisions 12-3
12.3 Collision Analysis Examples in One Dimension 12-3
12.3.1 A Moving Vehicle into a Stopped Vehicle 12-3
12.3.2 Vehicles Colliding in the Same Direction 12-6
12.3.3 Vehicles Colliding in Opposite Directions 12-9
12.4 Collision Analysis in Two Dimensions 12-10
12.4.1 Graphical Analysis 12-11
12.4.1.1 Coordinate System 12-12
12.4.1.2 Vector Addition 12-12
12.4.1.3 Collision Analysis Using Vector Diagrams 12-16
12.4.1.4 Constructing a Vector Diagram 12-21
12.4.2 Mathematical Analysis 12-26
12.4.2.1 Derivation of Basic Equations 12-26
12.4.2.2 Derivation of the ■v and PDOF Equations 12-28

12.5 Evidence Required for COLM Analysis 12-34
12.5.1 Determining Pre- and Post-Impact Directions Using the Impact Circle . . . 12-34
12.5.2 Determining Vehicle Weights 12-37
12.5.3 Determining Post-Impact Speeds 12-37
12.6 Special Considerations and Limitations 12-38
12.6.1 Multiple Departure Analysis 12-38
12.6.2 Multiple Collisions 12-39
12.6.3 Low Speeds 12-41
12.6.4 Small Approach Angles 12-41
12.6.5 Large Weight Differences 12-42
12.6.6 Large Momentum Ratios 12-45
12.7 Summary 12-46

13 Work and Energy 13-1
13.1 Work 13-1
13.2 Mechanical Energy 13-6
13.2.1 Kinetic Energy 13-6
13.2.2 Potential Energy 13-8
13.3 Conservation of Energy 13-10
13.3.1 Systems 13-10
13.3.2 Conservative Forces 13-11
13.3.3 Non-Conservative Forces 13-12
13.4 Work–Energy Theorem 13-14
13.5 Derivation of the Kinetic Energy Formula 13-15
13.6 Power 13-17

14 Rotational Mechanics 14-1
14.1 Uniform Circular Motion 14-1
14.2 Lateral Acceleration 14-3
14.3 Rotational Motion 14-7
14.3.1 Angular Displacement 14-8
14.3.2 Angular Velocity 14-8
14.3.3 Angular Acceleration 14-8
14.4 Mass Moment of Inertia 14-9
14.4.1 The Parallel Axis Theorem 14-12

14.4.2 Radius of Gyration 14-14
14.5 Newton's Second Law for Rotation 14-14
14.6 Changing Torque and Gear Ratios 14-15
14.7 Rotational Kinetic Energy 14-19
14.8 Angular Momentum 14-21
14.9 Eccentric Collision Analysis Using Rotational Mechanics 14-31
14.9.1 Derivation of Δv from Rotational Mechanics Concepts 14-32
14.9.2 The Work–Energy Theorem for Rotation 14-34
14.9.3 Computing Impact Speed from Δv 14-37

III Laboratory Exercises 14-47

15 Laboratory Policies 15-1
15.1 Safety 15-1
15.2 Laboratory Schedule 15-1

16 Motor Control and Speed Sensing 16-1
16.1 Introduction 16-1
16.2 Assignment 16-1
16.3 Proceedure 16-2
16.4 Reporting Requirements 16-8

17 Determining Inertial Properties 17-1
17.1 Problem Statement 17-1
17.2 Problem Solving Strategy 17-1
17.3 Reporting Requirements 17-2

18 Jumping Impulse 18-1
18.1 Objective 18-1
18.1.1 Calibration Using Multiple Regression 18-1
18.1.2 Measure Force and Determine Impulse 18-1
18.2 Theory 18-1
18.2.1 Determining Jump Height Based on Impulse 18-1
18.2.2 Jump Height from Hang Time 18-4
18.2.3 Multiple Regression for Calibration 18-5

18.3 Procedure 18-6
18.3.1 Set Up LabVIEW for Data Acquisition 18-7
18.3.2 Determine Gain Constants 18-7
18.3.3 Measure Impulse and Height 18-9
18.3.4 Data Analysis 18-11
18.4 Reporting Requirements 18-11

19 Balancing Rotating Masses 19-1
19.1 Objective 19-1
19.2 Theory 19-1
19.3 Procedure 19-3
19.3.1 Establish Smooth Baseline 19-3
19.3.2 Establish Unbalanced Baseline 19-4
19.3.3 Force Balance 19-4
19.3.4 Moment Balance 19-4
19.4 Reporting Requirements 19-5

20 Free Vibration 20-1
20.1 Objective 20-1
20.2 Theory 20-1
20.2.1 Differential Equation of Motion 20-1
20.2.2 Log Decrement 20-4
20.2.3 Indirect Weighing 20-5
20.3 Procedure 20-5
20.4 Reporting Requirements 20-7

21 Forced Vibration 21-1
21.1 Objective 21-1
21.2 Theory 21-1
21.3 Procedure 21-3
21.4 Reporting Requirements 21-3

22 Small Engine Rebuild 22-1
22.1 Objective 22-1
22.2 Tasks 22-1
22.3 Dissassembly 22-2

22.4 Reporting Sheet for Engine Rebuild Lab 22-10

23 Crash Analysis and Deposition Exercise 23-1

23.1 Objective 23-1
23.2 The Crash Scenario 23-1
23.3 Supporting Documents 23-5
23.3.1 Computer Generated Drawings 23-5
23.3.2 Friction Determination 23-7
23.3.3 Vehicle Information 23-8
23.4 Assignment 23-9
23.5 Ethics for Accident Investigation and Reconstruction 23-9
23.6 Reporting Requirements 23-10

24 Ansys Workbench Exercise 24-1

24.1 Objective 24-1
24.2 Theory and Hand Calculations 24-1
24.3 Ansys Exercise 24-18

25 Rocket Lab Practical 25-1

25.1 Objective 25-1
25.2 Background and Theory 25-2
25.2.1 Thrust Calculation 25-2
25.2.2 Mass Moment of Inertia 25-5
25.2.3 Angular Velocity Prediction 25-7
25.3 Example Numerical Integration for a Simple Pendulum 25-8
25.4 LabVIEW Data Acquisition 25-8
25.5 Instrument Hookup and Calibration 25-9
25.5.1 Accelerometer 25-9
25.5.2 Pressure Transducer 25-11
25.5.3 LabVIEW Programming 25-11
25.5.4 Quadrature Encoder 25-11
25.6 Procedure 25-13

IV Supplemental Material **25-15**

A Grading Sheet for Technical Reports **A-1**

B Participation Survey **B-1**

C Formal Letter Template **C-1**

Bibliography **C-2**

Syllabus

Instructor: Dr. Jeremy S. Daily

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Office: 2080 Stephenson

Office Hours: Right after class on M and W. Otherwise, drop in or schedule an appointment (e-mail or phone)

Classroom: U1

Laboratory Stephenson 1085

Lecture Time: 2:00-3:15 PM, Mondays and Wednesdays

Lab Time: 2:00-5:00 PM, Tuesdays and Thursdays

This course notebook is required for the course and can only be purchased from Dr. Daily. While the pages in here are designed to help you take notes, additional writing space will be required. Therefore, loose leaf paper is recommended to augment the notebook.

This notebook can be accessed in electronic form at

http://personal.utulsa.edu/∼jeremy-daily/ME4024/MachineDynamicsCourseNotebook.pdf

A website dedicated to Machine Dynamics is located at

http://personal.utulsa.edu/∼jeremy-daily/ME4024/ME4024Syllabus.html

Engineering graph paper can be printed from

http://personal.utulsa.edu/∼jeremy-daily/downloads/EngineeringPaper.pdf

Course Bulletin Description

Kinematic and force analysis of machines and mechanisms. Mechanical vibrations, balancing, and critical speed. Dynamic measurement using transducers and data acquisition systems, analysis and interpretation of data, lab report writing. Introduction to multi-body simulation using modern engineering software. Written laboratory reports. Three hours lecture and three hours laboratory per week. Prerequisite: ES 2023 - Dynamics

This required four-credit hour course is offered once a year, typically at the end (spring semester) of the junior year.

Dynamic Course Outline

The following is a dynamically updated schedule for the class. It is a shared Google calendar

http://www.google.com/calendar (Search for ME 4024)

and can be integrated into your own personal calendaring system. While the calendar can be updated and changed as the course goes on, the schedule will remain fairly rigid during the semester. All changes and details concerning specific events and items on the schedule will be updated through the Google calendar for this course.

The calendar ID is i26vok4v83gtp2ckn7b64em95o@group.calendar.google.com

Course Policies

Text Books

Reference (not required)

- Fundamentals of Traffic Crash Reconstruction by J.G. Daily, N. Shigemura, and J.S. Daily, Institute of Police Technology and Management, 2006, ISBN 1-884566-63-4
- Engineering Vibration, 3rd Edition by D.J. Inman, Prentice Hall, 2008, ISBN 0-13-228173-2
- Mechanics of Machines by W.L. Cleghorn, Oxford University Press, 2005, ISBN 0-19-515452-5

- Theory of Machines and Mechanisms, 4th Edition by J.J. Uicker, G.R. Pennock, and J.E. Shigley, Oxford University Press, 2011, ISBN 0-19-537123-9
- Any sophomore level dynamics book.

Grading Procedures

A separate sheet will be provided to explain grade allocations.

90-100 = A, 80-89 = B, 70-79 = C, 60-69 = D, < 60 = F

The instructor reserves the right to lower the minimum requirements for each letter grade.

Exam Policy

Exams are open book and open notes; closed computer.

Computer Usage

Matlab, Mathematica, SolidWorks, LabVIEW, ANSYS and other specialty software will be used for labs and homework. These programs are available in the Shared Undergraduate Computer lab and the Machine Dynamics Lab.

Late Submission and Absences

Late submission of homework will receive no score. Late computer projects will receive no score. Exams have mandatory attendance. Make-up exams will be offered only under very exceptional circumstances provided prior permission from the instructor is obtained. Neatness and clarity of presentation will be given due consideration while grading homework and exams.

Class Conduct

Please do whatever necessary to maintain a friendly, pleasant and business-like environment so that it will be a positive learning experience for everyone. Please turn off all cell phone ringers or any other device that could spontaneously make noise.

Academic Misconduct

All students are expected to practice and display a high level of personal and professional integrity. During examinations each student should conduct himself in a way that avoids even the appearance of cheating. Any homework or computer problem must be entirely the students' own work. Consultation with other students is acceptable; however copying homework from one another will be considered academic misconduct. Any academic misconduct will be dealt with under the policies of the College of Engineering and Natural Sciences. This could mean a failing grade and/or dismissal. The policy of the University regarding withdrawals and incompletes will be strictly adhered to.

Center for Student Academic Support

Students with disabilities should contact the Center for Student Academic Support to self-identify their needs in order to facilitate their rights under the Americans with Disabilities Act. The center for Student Academic Support is located in Lorton Hall, Room 210. All students are encouraged to familiarize themselves with and take advantage of services provided by the Center for Student Academic Support such as tutoring, academic counseling, and developing study skills. The Center for Student Academic Support provides confidential consultations to any student with academic concerns as well as to students with disabilities.

The University of Tulsa Mission

The University of Tulsa is a private, independent, doctoral-degree-granting institution whose mission reflects these core values: excellence in scholarship, dedication to free inquiry, integrity of character, and commitment to humanity. The university achieves its mission by educating men and women of diverse backgrounds and cultures to become literate in the sciences, humanities, and arts; think critically, and write and speak clearly; succeed in their professions and careers; behave ethically in all aspects of their lives; welcome the responsibility of citizenship and service in a changing world; and acquire the skills and appetite for lifelong learning.

While one course cannot accomplish the mission of the University experience, ME4024 does emphasize the following aspects of the University's mission:

- Clear writing is practiced by submitting technical laboratory reports following the laboratory experiment. Also, neat and clean homework assignments must be submitted in a timely manner.
- The understanding of dynamic processes is a fundamental skill required for success in an engineering career. Also, the ability to interact with modern measurement systems is a necessary skill for success.
- The objective reporting of experimental data is paramount and fundamental to engineering ethics.
- The ability to learn how to use new software and hardware systems is critical to being able to maintain the appetite for lifelong learning in a dynamic engineering environment.

Part I

Lecture

1 Dynamics Problem Solving

Machine Dynamics is a full and rich subject for study. Concepts in machine dynamics apply to all industries that employ moving parts. The goal of this course is to develop the skills and problem solving strategies to understand these concepts in both industry and further study. This chapter should solidify techniques and methods taught in previous dynamics courses and develop fundamental concepts in problem solving with dynamic systems.

1.1 Dynamics Vocabulary

Fill in the appropriate definitions for the following terms.

Mechanics:

Statics:

Dynamics:

Kinematics:

Kinetics:

Machines:

Control:

1.2 Problem Solving

The following steps are necessary when trying to solve machine dynamics problems

1. Read the problem carefully and understand what it is asking. Take note of units.

2. Draw large diagrams and carefully tabulate data. This includes a Free Body Diagram (FBD) and an Inertial Response Diagram (IRD).

3. Establish the appropriate coordinate systems and a transformation between them.

4. Solve the problem using symbols as far as possible using relevant kinetic and kinematic principles.

5. Write a well commented computer program to obtain numerical and graphical results.

6. Check to see if the the solution makes sense.

7. Solve the problem using another technique and try to get the same answer.

All work should be on engineering paper. The header should contain the course number/course name, your name, the project/assignment, and page numbers. No markings should be outside the work area. All diagrams and sketches should be centered, make use of a large portion of the page, and no work should be shown on the side of the drawings. All work must be neat and orderly with written explanations for the procedures. The final answer should be boxed with the appropriate significant figures.

1.3 Vector Operations

Vectors are a fundamental mathematical tool that are used to describe physical pheneomenon.

1.3.1 Cartesian Coordinates

$$\vec{A} =$$

$\hat{i}, \hat{j}, \hat{k}...$

Magnitude of $\vec{A}$:

Direction of $\vec{A}$:

Direction Cosines:

1.3.2 Dot Product

$$\vec{A} \cdot \vec{B} =$$

or

$$\vec{A} \cdot \vec{B} =$$

or

$$\vec{C} = \vec{A} \times \vec{B} = \begin{vmatrix} & & & & \\ & & & & \\ & & & & \end{vmatrix}$$

$$\vec{A} \cdot \vec{B} =$$

1.3.3 Cross Product

$$\vec{C} = \vec{A} \times \vec{B} = \begin{vmatrix} & & & & \\ & & & & \\ & & & & \end{vmatrix}$$

Expand the determinate to compute $\vec{C}$

1.3.4 Product Rule

$$\frac{d(a\vec{B})}{dt} =$$

$$\frac{d(\vec{A} \cdot \vec{B})}{dt} =$$

$$\frac{d(\vec{A} \times \vec{B})}{dt} =$$

1.3.5 Partial and Total Derivatives

Given a function of n variables that are functions of time:

$$f = f(x_1, x_2, \cdots, x_n, \dot{x}_1, \dot{x}_2, \cdots, \dot{x}_n, t)$$

where each $x_i = x_i(t)$.

$$\frac{df}{dt} =$$

if t does not explicitly appear in f, then

1.3.6 Time Derivative of a Vector

$$\frac{d}{dt}(\vec{A}) = \dot{\vec{A}} =$$

1.4 Kinematic Relationships for Rectilinear Motion

1.4.1 Velocity, distance, and time

1.4.2 Acceleration, velocity and time

1.4.3 Displacement, velocity and acceleration

1.4.4 Special Cases for Constant Acceleration

1.
2.
3.

Drag Factor is a ratio of accelerations when slowing:

$$f = -\frac{a}{g}$$

Acceleration Factor is a ratio of accelerations when speeding up:

$$f = \frac{a}{g}$$

1.5 Newton’s Laws of Motion

1.

2.

3.

1.6 Inertial Reference Frames

1.7 Force, Mass and Weight

Force:

1.

2.

3.

Mass:

Weight:

1.

2.

3.

How much does a 4200-lb car weigh in the SI system?

SI Units

US Units

1.8 Example

Two masses A and B are hung from a pulley and released from rest. Determine the velocity of A after 2 seconds. Also, determine the velocity of *A* after 8 feet. Mass A weighs 250 lb and mass B weighs 150 lb.

Ignore the moment of inertia of the pulley, the friction in the bearings, the mass of the cable, and the extension in the cable.

Determine the values of mass.

Draw Kinetic Diagrams (Inertial Response Diagrams):

Write the Governing Equations:

A:

B:

Now what if $w_A = 1250$ lb and $w_B = 1150$ lb?

Solve using work and energy.

1.9 Tangent and Normal Coordinates

1.10 Radial and Transverse Coordinates

1.11 Homework Problem Set 1

1. A vehicle begins skidding on a surface with a drag factor of 0.72. It skids for 24 m and then skids 19 m across another surface with a drag factor of 0.50. It then impacts a tree at 48 kph and stops. The Perception-Response Time (PRT) of the driver/vehicle is determined to be about 1.8 seconds.

   a) What is the initial speed of the vehicle? **95.42 kph**

   b) What is the PRT distance? **47.71 m**

   c) What is the total distance to impact, starting at the perception-response (PR) point? **90.71 m**

   d) What is the distance from the PR point to the beginning of the second skid? **71.71 m**

   e) If there is no impact with the tree, what is the speed at the beginning of the last skid? **49.12 kph**

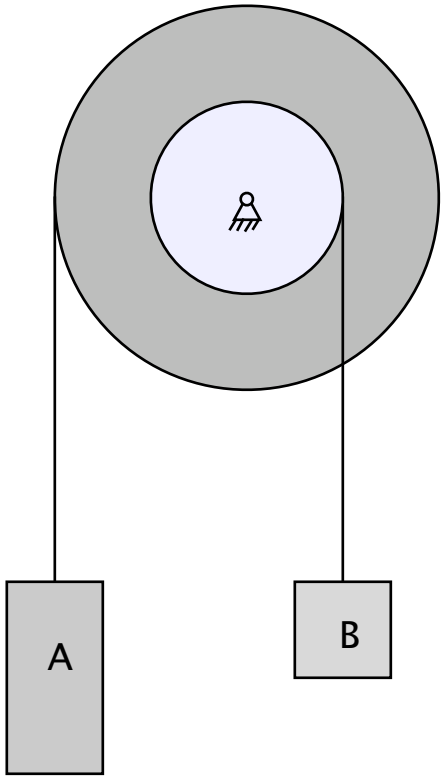
   f) What is the maximum speed the vehicle could be going to stop at the tree without impact, using the initial PR point as a reference? (Hint: The distance of the first skid is now unknown.) **87.01 kph**

   g) What is the skid distance across the first surface? **28.20 m**

2. The two weights (A and B) are initially at rest. Determine the angular velocity (magnitude and direction) of the pulley 1 second after letting go. The inner diameter is 15 cm and the outside diameter is 25 cm. Weight A is 10 kg and weight B is 8 kg. Assume the chord is inextensible and has negligible mass.

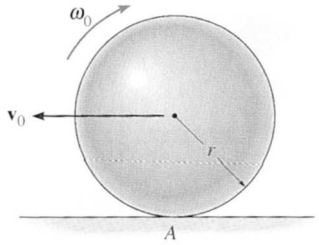
   a) Solve neglecting the moment of inertia of the pulley. (Ans: $\omega = 31.68$ rad/sec)

   b) Solve including the moment of inertia of the pulley when its radius of gyration is 20 cm and it weighs 30.6 kg. (Ans: $\omega = 4.48$ rad/sec)


A
B

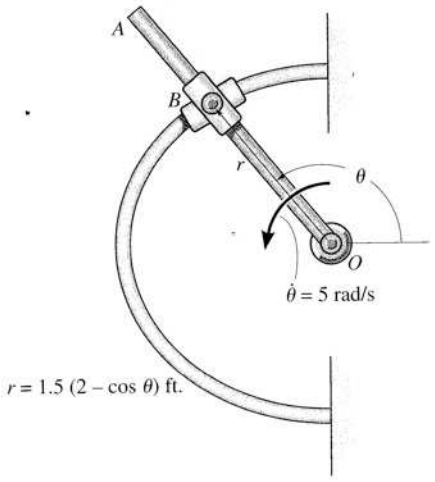
3. Do the following problem from dynamics.

19-29. If the ball has a weight W and radius r and is thrown onto a *rough surface* with a velocity $\mathbf{v}_0$ parallel to the surface, determine the amount of backspin, $\boldsymbol{\omega}_0$, it must be given so that it stops spinning at the same instant that its forward velocity is zero. It is not necessary to know the coefficient of friction at A for the calculation.



4. Do the following problem from dynamics.

13-89. Rod OA rotates counterclockwise with a constant angular velocity of $\dot{\theta} = 5$ rad/s. The double collar B is pin-connected together such that one collar slides over the rotating rod and the other slides over the *horizontal* curved rod, of which the shape is described by the equation $r = 1.5(2 - \cos\theta)$ ft. If both collars weigh 0.75 lb, determine the normal force which the curved rod exerts on one collar at the instant $\theta = 120°$. Neglect friction.



Prob. 13–89

1.12 Momentum

Newton’s 2nd Law of Motion for Rigid Bodies

Translation:

Rotation:

where $\sum \vec{F} \ldots$

$\vec{L} \ldots$

$\vec{H}_G \ldots$

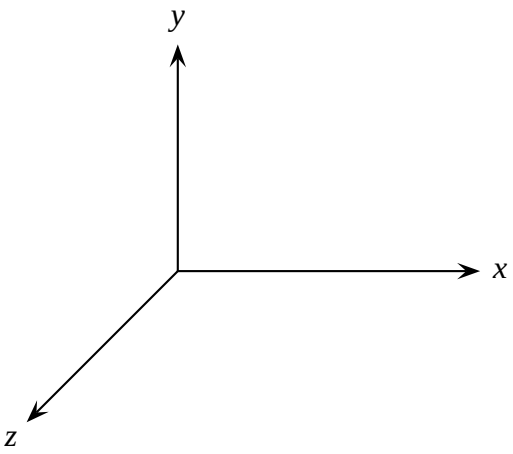
$$\vec{H}_G = \left\{ \quad \right\} = \left[\quad \right] \left\{ \quad \right\}$$

Angular Momentum Vector =

So, what does this mean?

1.

2.


y
x
z

1.12.1 Concept of Impulse

1.12.2 System Momenta

In Words:

X :

Y :

θ :

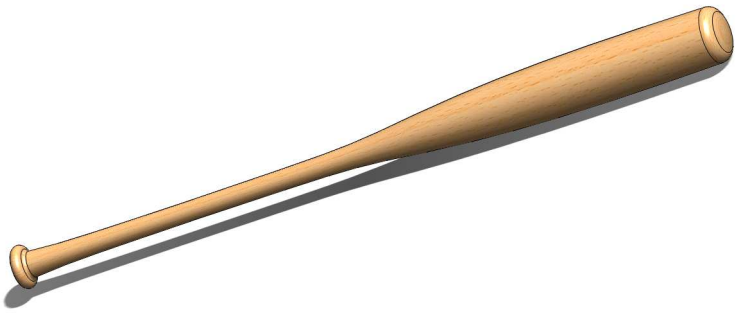
1.12.3 Moment of Inertia

From Wikipedia:

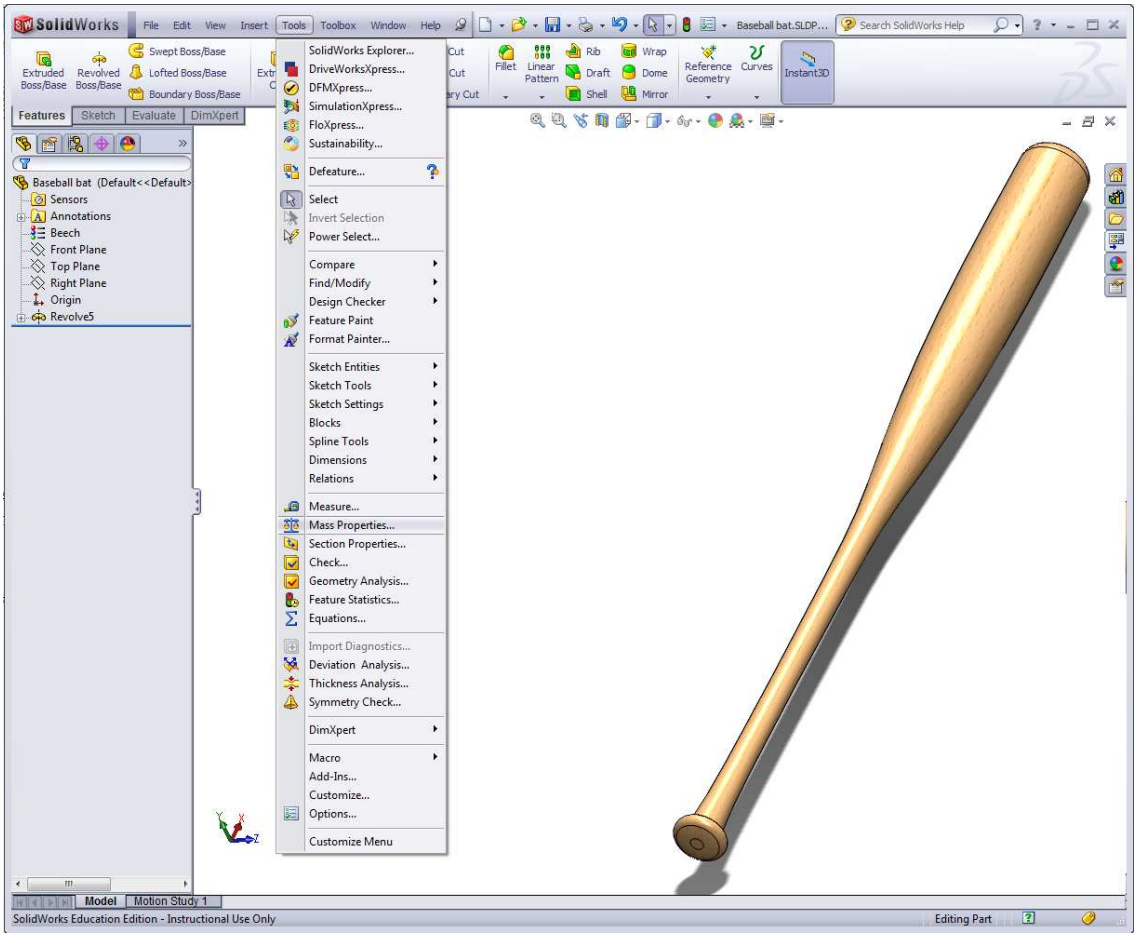
> A baseball bat is a smooth wooden or metal club used in the game of baseball to hit the ball after the ball is thrown by the pitcher. It is no more than 2.75 inches in diameter at the thickest part and no more than 42 inches (1,100 mm) long. It typically weighs no more than 33 ounces (0.94 kg), but it can be different from player to player.

Baseball bat swing is easier when the bat is backwards, because

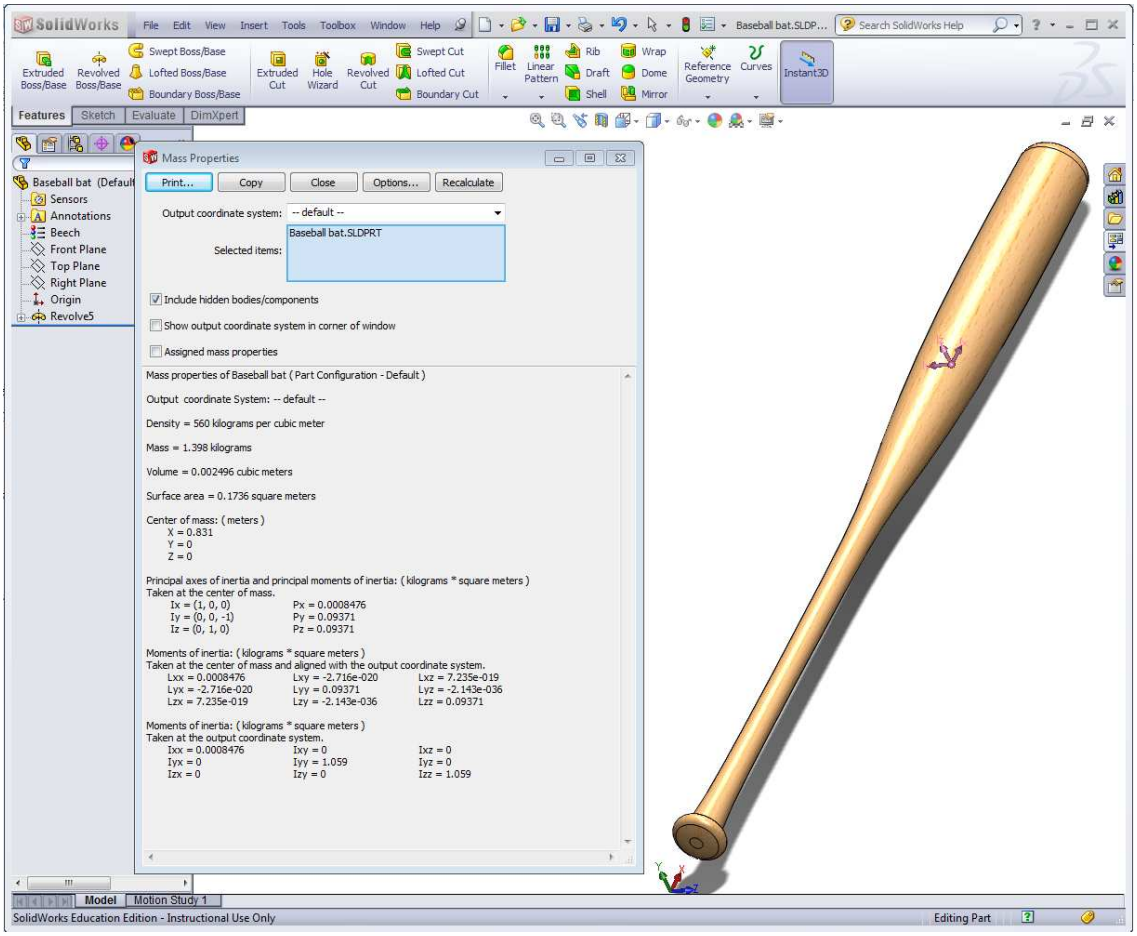
The following Solidworks model was downloaded from http://grabcad.com/home



Let's see what Solidworks thinks the mass properties are. Click on the Tools > Mass Properties menu.



The dialog box shows


SolidWorks
Mass Properties
Print...
Copy
Close
Options...
Recalculate
Output coordinate system: -- default --
Selected items: Baseball bat.SLDPRT
Include hidden bodies/components
Show output coordinate system in corner of window
Assigned mass properties
Mass properties of Baseball bat (Part Configuration - Default)
Output coordinate System: -- default --
Density = 560 kilograms per cubic meter
Mass = 1.398 kilograms
Volume = 0.002496 cubic meters
Surface area = 0.1736 square meters
Center of mass: (meters)
X = 0.831
Y = 0
Z = 0
Principal axes of inertia and principal moments of inertia: (kilograms * square meters)
Taken at the center of mass.
Ix = (1, 0, 0) Px = 0.0008476
Iy = (0, 0, -1) Py = 0.09371
Iz = (0, 1, 0) Pz = 0.09371
Moments of inertia: (kilograms * square meters)
Taken at the center of mass and aligned with the output coordinate system.
Lxx = 0.0008476 Lxy = -2.716e-020 Lxz = 7.235e-019
Lyx = -2.716e-020 Lyy = 0.09371 Lyz = -2.143e-036
Lzx = 7.235e-019 Lzy = -2.143e-036 Lzz = 0.09371
Moments of inertia: (kilograms * square meters)
Taken at the output coordinate system.
Ixx = 0.0008476 Ixy = 0 Ixz = 0
Iyx = 0 Iyy = 1.059 Iyz = 0
Izx = 0 Izy = 0 Izz = 1.059
Model
Motion Study 1
SolidWorks Education Edition - Instructional Use Only
Editing Part

The print dialog gives:

Mass properties of Baseball bat (Part Configuration - Default)

Output coordinate System: -- default --

Density = 560 kilograms per cubic meter

Mass = 1.398 kilograms

Volume = 0.002496 cubic meters

Surface area = 0.1736 square meters

Center of mass: (meters)
X = 0.831
Y = 0
Z = 0

Principal axes of inertia and principal moments of inertia: (kilograms * square meters)
Taken at the center of mass.
Ix = (1, 0, 0) Px = 0.0008476
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Moments of inertia: (kilograms * square meters)
Taken at the output coordinate system.
Ixx = 0.0008476 Ixy = 0 Ixz = 0
Iyx = 0 Iyy = 1.059 Iyz = 0
Izx = 0 Izy = 0 Izz = 1.059

How can we verify these numbers. Information is cheap and easy with the internet. Evalating information is what an engineer should be able to do.

1. Verify weight using a measurement
2. Doublecheck dimensions (e.g solid vs hollow tubes)
3. Adjust denisty so

$$\rho = \frac{\qquad}{\qquad}$$

4. Calculate moment of inertia of simple shapes.

5. Use Parallel Axis Theorem

Inertia is a tensor

Radius of Gyration

Vibration Measurements

1.13 Collisions in 2-D

1.13.1 Center of Percussion

Impacts at the center-of-percussion result in zero net force at the pivot point, this location has long been identified with the sweet spot.

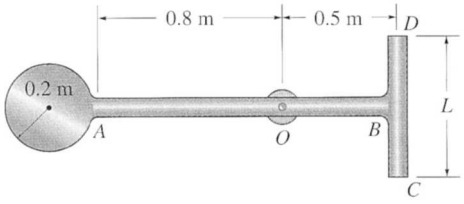
1.13.2 Collision Reconstruction Example Using The Conservation of Linear Momentum

1.13.3 Collision Reconstruction Example Using Angular Momentum

1.14 Homework Problem Set 2

1. Do the following problem from dynamics. The radius of the disk A is 0.2 m. Report your results in slug-ft^2.

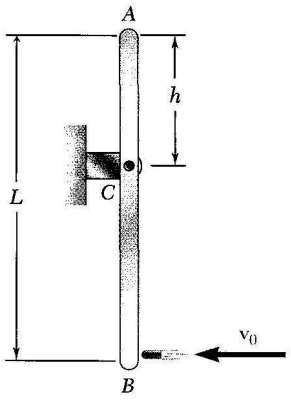
17-14. The assembly consists of a disk having a mass of 6 kg and slender rods AB and DC which have a mass of 2 kg/m. If $L = 0.75$ m, determine the moment of inertia of the assembly about an axis perpendicular to the page and passing through O.



2. A 40-g bullet is fired with a horizontal velocity of 600 m/s into the lower end of a slender 7-kg bar of length $L = 600$ mm. Knowing that $h = 240$ mm and that the bar is initially at rest, determine

   a) the angular velocity of the bar immediately after the bullet becomes embedded,

   b) the impulsive reaction at C, assuming that the bullet becomes embedded in 0.001 seconds.



3. A Chevrolet Caprice weighing 3600 lb and a Ford Crown Victoria weighing 3800 lb are involved in a collision. The Chevrolet may be assumed to be on the *x*-axis with an approach of 0°. The approach of the Ford is at 80°. The departure of the Chevy is 30° and the departure of the Ford is 45°. The departure speed of the Chevy is 30 mph and the departure speed of the Ford is 35 mph. Assume the friction forces are small compared to the collision forces (i.e. assume no external impulse).

   a) What is the speed of the Ford at impact?

   b) What is the speed of the Chevy at impact?

   c) What is the Δv of the Ford in the collision?

   d) What is the Δv of the Chevy in the collision?

   e) Determine the value of the collision impulse assuming the duration of the impact is 0.150 seconds.

   f) Now test your assumption. If the coefficient of friction is 0.7 and the duration of the impact is 0.150 seconds, estimate the impulse from the friction force during the collision. Compare this value to the collision impulse.

   g) What is the angle of the Principal Direction Of Force (PDOF) for the Ford?

   h) What is the angle of the PDOF for the Chevy?

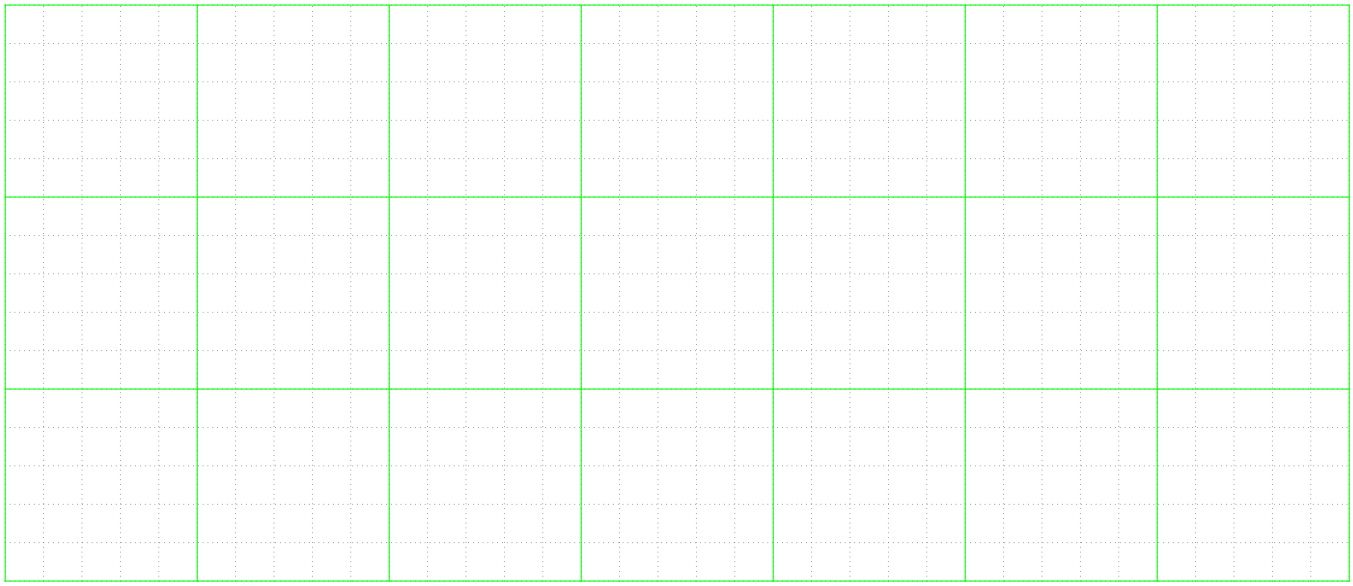
   i) Are the PDOF angles opposite in direction? If so, why? If not, why not?

   j) How much energy is lost in the collision phase of this accident?

1.15 Work and Energy

1.15.1 Kinetic Energy

Consider a mass translating and rotating in a plane.



Kinetic Energy from Translation:

$$T =$$

Kinetic Energy from Rotation:

$$T =$$

Kinetic Energy about point O:

1.15.2 Potential Energy

Gravity

Elastic Strain Energy

1.15.3 Work

1.15.3.1 Conservative Forces

Work of a Weight

Work of a spring

1.15.3.2 Non-Conservative Forces

Friction

In General

1.15.3.3 Work-less Forces

Constraints:

1.15.4 The Work-Energy Theorem

Energy is the ability to do work.

In General:

The work from friction:

1.15.5 The Conservation of Mechanical Energy

For Conservative Forces we get the Conservation of Energy:

Example: Charpy Impact Machine

Example: A 10-kg rod has its ends constrained to move in a vertical or horizontal slot. A spring is attached to the end in the vertical slot and has a stiffness of 800 N/m. It is not stretched when the rod is horizontal. The rod is 0.4 m long and its center of gravity is in the middle. Find static equilibrium and find the angular velocity when the rod is released from 30 degrees. Use Newton's Laws to develop the equation of motion.

1.16 Homework Problem Set 3

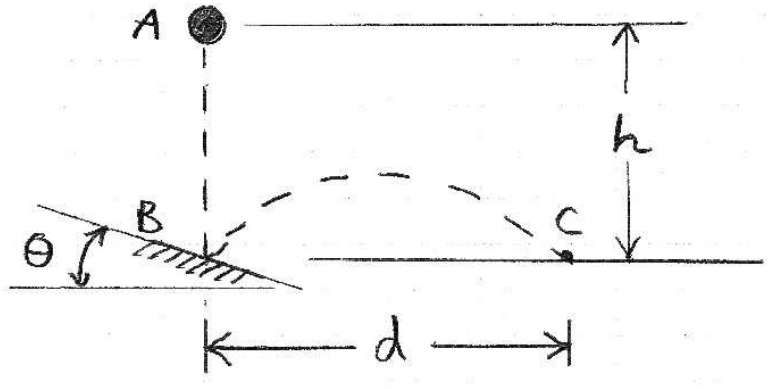
1. Using the Work-Energy Theorem, show that the speed calculation based on the distance of a locked wheel skid is independent of the vehicle weight. This will require you to derive a slide to stop equation based on work done by friction and the kinetic energy of the vehicle. The speed equation is

   $$S = \sqrt{30df}$$

   where S is in mph, d is in ft, and f is the drag factor.

2. A 30-lb uniform rod is released from rest when it is in the near vertical position. It is allowed to fall freely. Determine the angle at which the bottom end starts to lift off the ground. Neglect friction at the bottom and the rod is 10 ft long.

3. A ball is dropped from Point A to plate B and bounces to point C. For $\theta = 20^\circ$ and a coefficient of restitution of 0.40, determine the distance, d, as a function of the height, h.



4. The system is released from rest. Knowing the energy dissipated in the axle friction is 10 J and the inertia of the pulley is negligible, determine

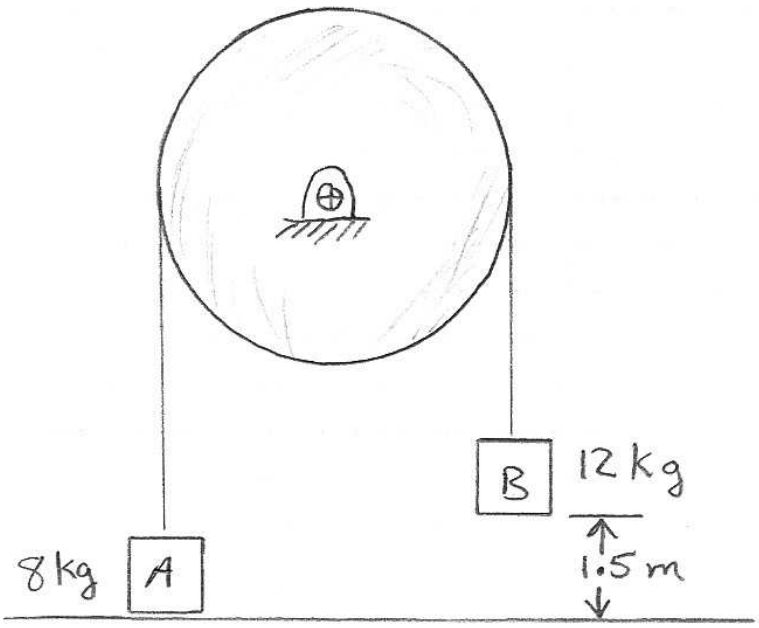
   a) the velocity of B as it hits the ground and

   b) the tension of the cable on each block.

   Now consider the pulley has a moment of inertia about its center of 4 kg-m^2 and a radius of 1m, determine

   c) the velocity of B as it hits the ground and

   d) the force from of the cable on each block.



1.17 Principle of Virtual Work

1.18 Lagrange Equations

Example: The Equations of Motion of a Pendulum

1.19 Simulation

1.19.1 State-Space Form

1.19.2 Euler's Method

1.19.3 Runge-Kutta

1.20 Homework Problem Set 4

Simulate 15 seconds of a double rod pendulum falling from a horizontal position. Assume frictionless bearings and the mass and moment of intertia each have a value of 1. You will need to derive the equations of motion first, then put the equations in state-space form. Once in state space form, you will need to write a computer program to simulate the motion of the pendulum. Turn in your computer code and a plot of the locus of the trace point, P on an X-Y plot with equal axes.

2 Vibration

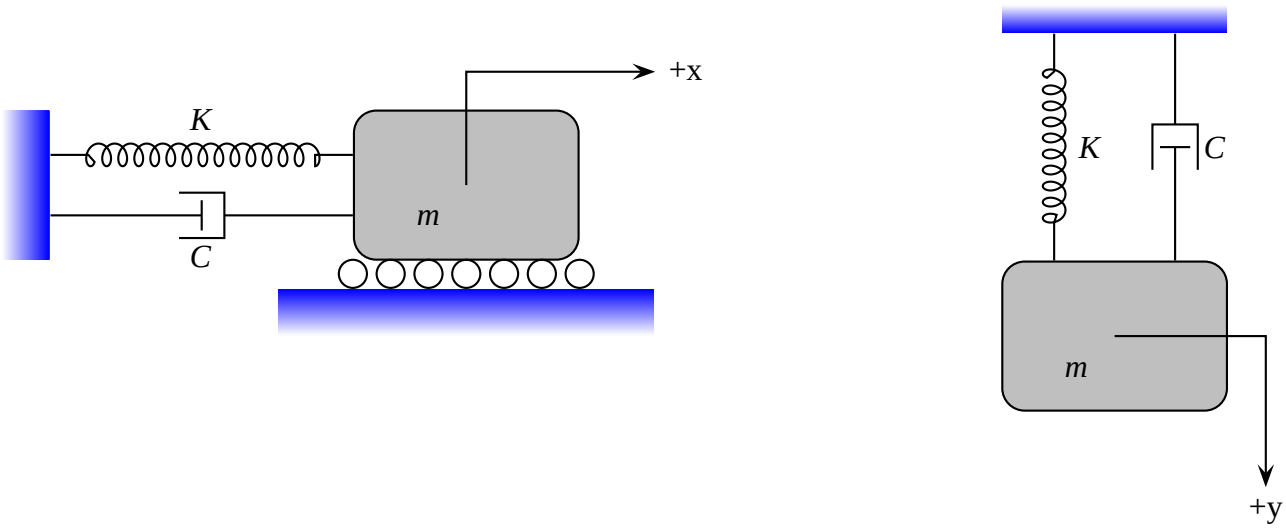
2.1 Free Vibration

2.1.1 Undamped Systems

2.1.2 Systems with Damping

2.2 Homework Problem Set 5

1. Show that the two systems below have the same equation of motion (in the presence of gravity).



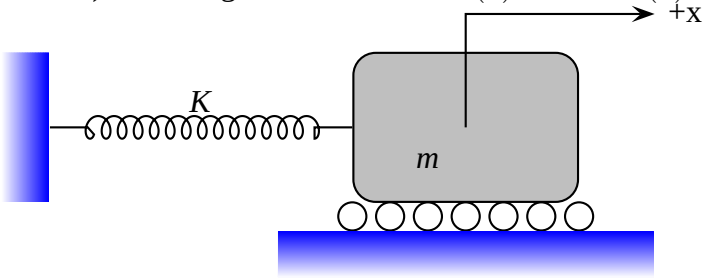
2. Evaluate A, ω_n, and ϕ and plot the responses of at least 3 periods with a computer.

$$x(t) = A\cos(\omega_n t + \phi)$$

a) $m = 5$ kg, $K = 2000$ N/m, $x(0) = 5$ cm, $v(0) = 0$

b) $m = 10$ kg, $K = 1000$ N/m, $x(0) = 5$ cm, $v(0) = 0$

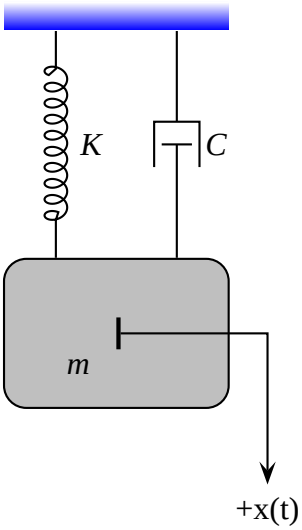
c) $m = 2$ kg, $K = 1000$ N/m, $x(0) = 5$ cm, $v(0) = 2$ cm/s



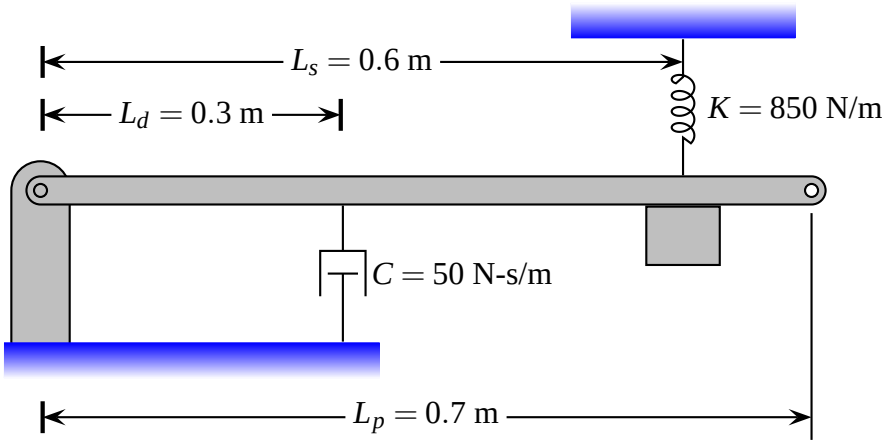
3. Evaluate A, ω_d, C, and ϕ for the system below. Plot the displacement responses with a computer with at least 4 periods. Compute the log-decrement, δ, based on ζ and compare that value to the log-decrement measured measured from the response.

a) $m = 5$ kg, $K = 2000$ N/m, $\zeta = 0.4$, $x(0) = 0$ cm, $v(0) = 5$ cm/s

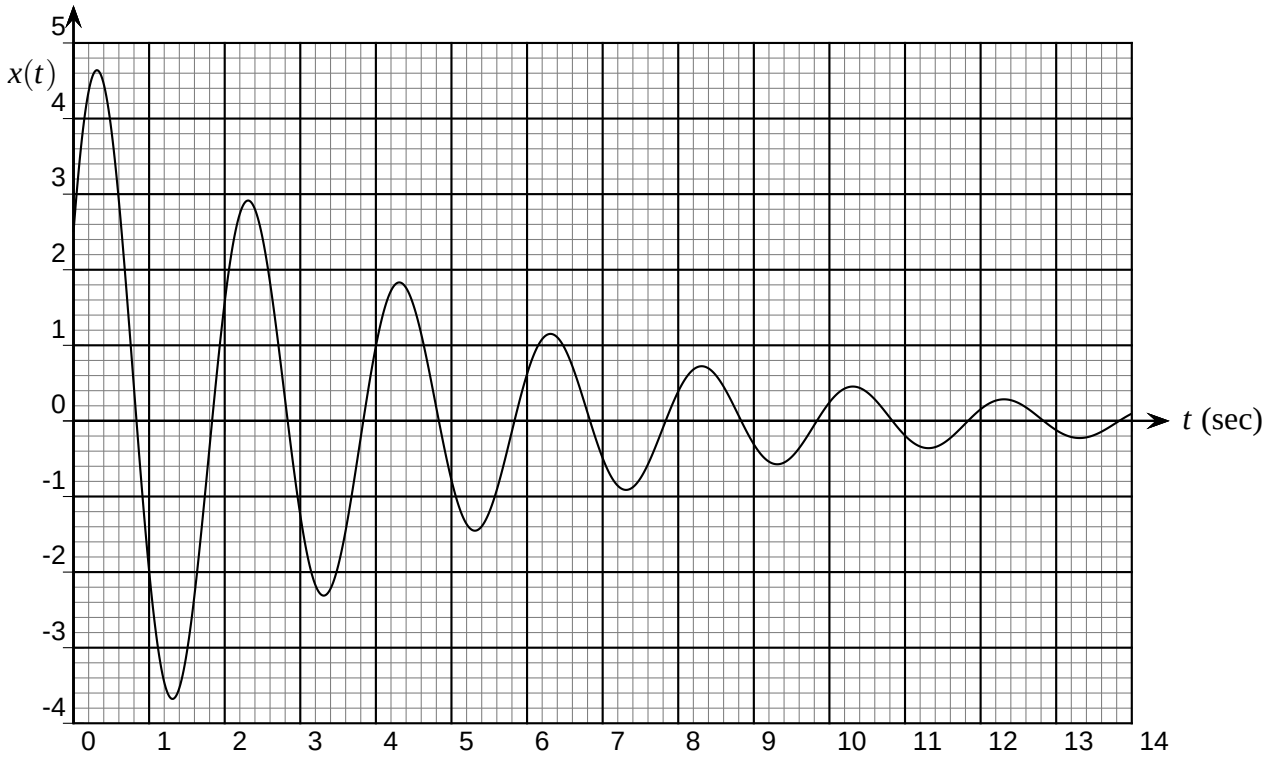
b) $m = 5$ kg, $K = 2000$ N/m, $\zeta = 0.2$, $x(0) = 2$ cm, $v(0) = 0$



4. Evaluate ζ, ω_d, and ω_n, for the following system when $I_o = 1.6$ kg-m^2 (about the point of rotation).



5. From the strip-chart data below, Determine ω_d, f_d, T, δ, ζ, ω_n, m, and C when $K = 200$ N/m.



2.3 Forced Vibration

2.3.1 Harmonic Forcing

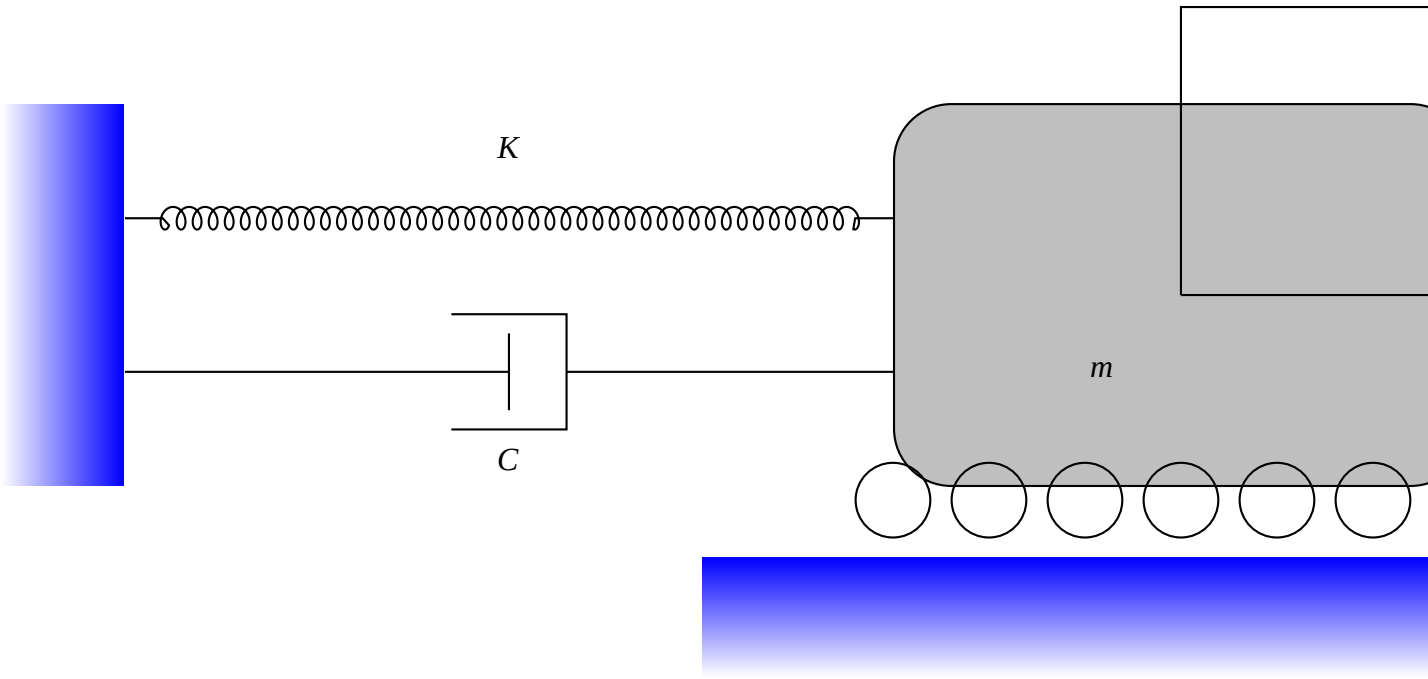
2.3.2 Forcing by Rotating Unbalance

2.3.3 Base Excitation

2.3.4 Frequency Response Functions

2.4 Homework Problem Set 6

1. Produce normalized graphs of the magnitude ratio and phase for the response of a system subject to a base excitation.
2. Produce normalized graphs of the magnitude ratio and phase for the response of a system subject to a rotating unbalance.
3. Consider the following steady-state forced vibration problem:



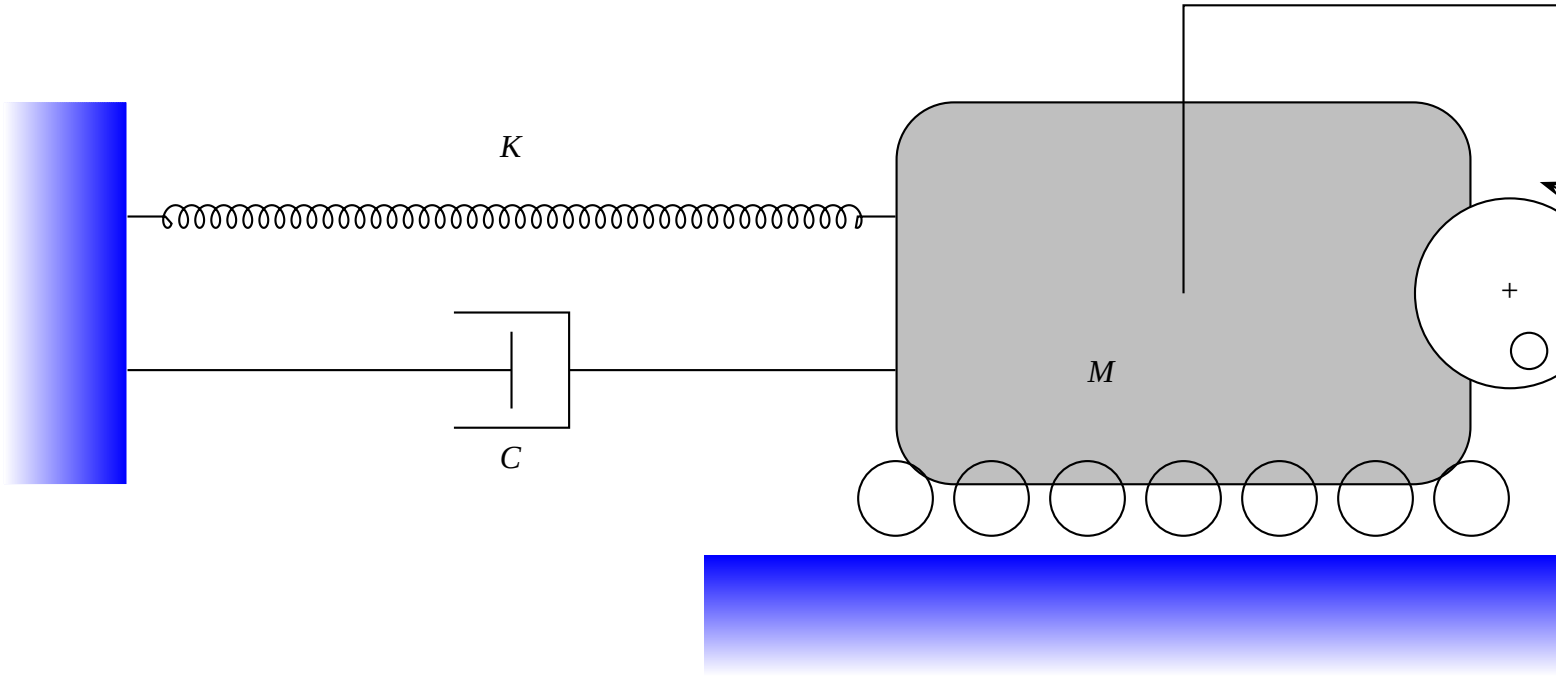
where: $m = 5$ kg, $k = 2000$ N/m, $c = 30$ N-s/m, $F = 20$ N, and $\omega = 15$ rad/s.

a) Write down the equation for the steady state response of the system, $x(t)$.

b) Plot exactly 3 periods of the steady-state solution.

c) Locate ϕ and $|H(\omega)|$ on the plots from problem 1. Compare the results from the graphs to your answer in part a.

4. Plot the amplitude of the response and the phase of the response based on of a rotating unbalance from $0 < \omega < 60$ rad/sec. $M = 5$ kg, $K = 2000$ N/m, $\zeta = 0.1$, $l = 4$ cm, and $m = 0.2$ kg.



The small dot represents a hole with a length from the center of rotation of l and a missing mass of m.

2.5 Critical Shaft Speed

2.5.1 Single Degree of Freedom Model

2.5.2 Rayleigh's Method

2.6 Homework Problem Set 7

1. Determine the eigenvalues and eigenvectors of the following matrix:

   a)

$$\begin{bmatrix} 8 & -12 \\ -12 & 1 \end{bmatrix}$$

   b)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 49 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 & 0 \\ 0 & 0 & 0 & 144 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

   c) What is the highest system frequency for the multiple degree of freedom system whose eigenvalues are determined in part b.

2. Determine the critical speed in RPM of an electric motor. The steel (E = 30e6 psi) shaft is 0.5 inches in diameter and the distance between the bearings is 12 inches. Consider the rotating element as being a single disk with a weight of 25 lb located midway between the bearings. Neglect the mass of the shaft and assume simple supports for the bearings. Is 1200 RPM a safe operating speed?

3. For the figure below, consult a text book for the static deflection and determine the critical speed in radians per second neglecting the mass of the shaft

   a) if the bearings act as simple supports

   b) the bearings act as cantilever supports (i.e. do not allow rotation)

   c) What do you suppose the actual critical speed of a real system would be? In other words, what if the previous boundary conditions are not exact.

4. The static deflection curve for a shaft supported in three bearings is shown below. The deflections and the corresponding weights are given. Find the lowest critical shaft speed using Rayleigh's Method.

5. Using Rayleigh's Method, show that the critical shaft speed for a simply supported uniform shaft is

$$\omega_n = 9.874\sqrt{\frac{EIg}{WL^3}}$$

where W is the total weight of the bar. The static deflection curve for a uniform horizontal shaft is

$$y = \frac{mgx}{24EI}(L^3 - 2Lx^2 + x^3)$$

where m is the mass per unit length.

3 Dynamic Force Analysis

- In machines that move...

- Quantification...

Recall,

Kinematics:

Kinetics:

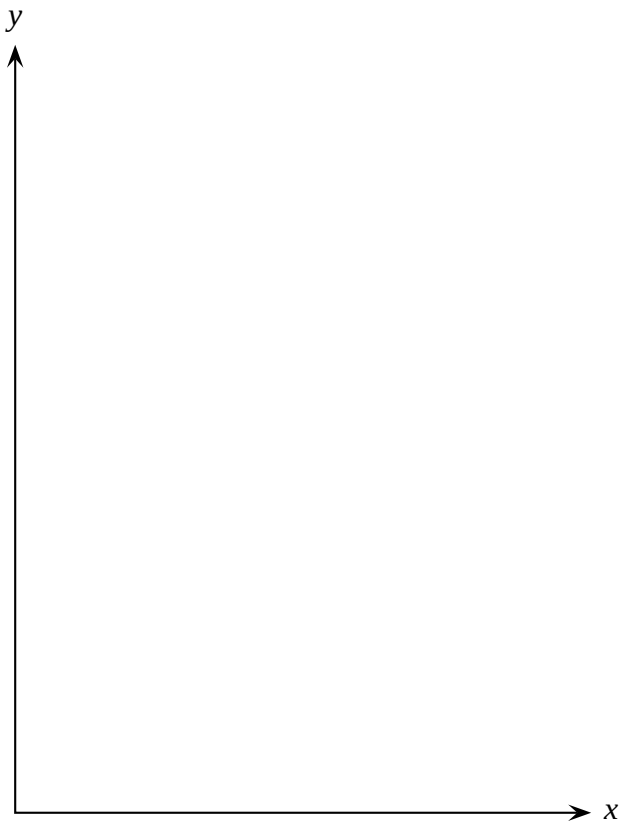
For Plane Motion:

3.1 D’Alembert’s Principle

Newton’s 2nd Law:

3.2

3.2.1 Example

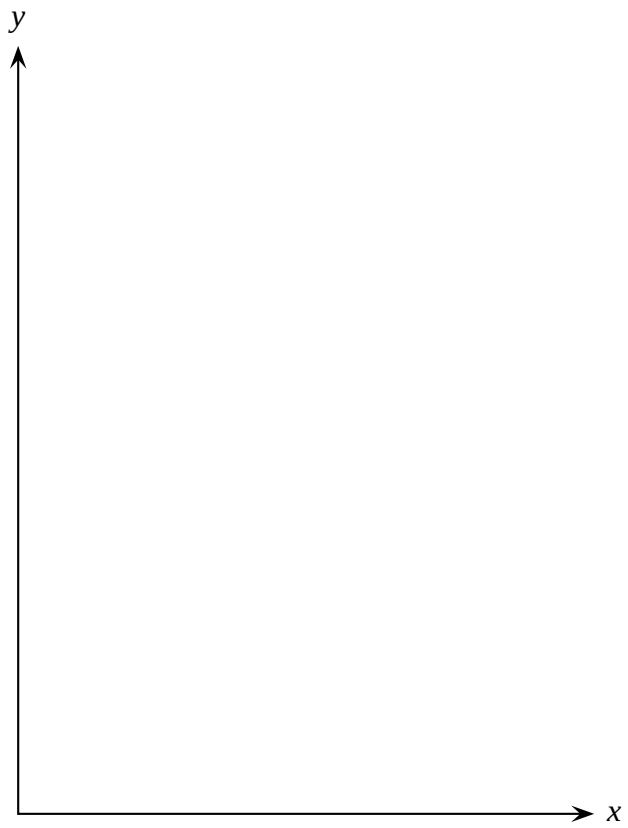


Relative Velocity Analysis

Absolute Velocity Analysis

Accelerations

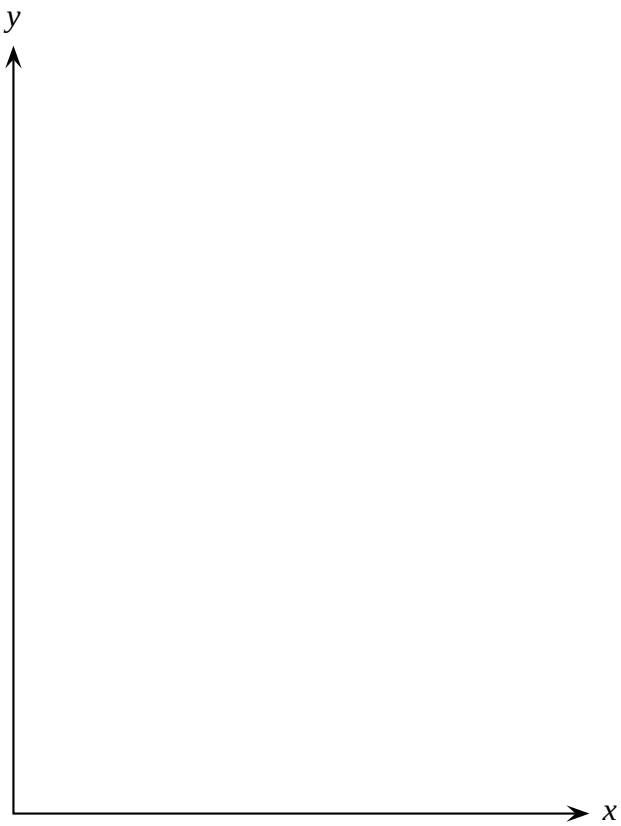
Use D’Alembert’s Principle:



Now, Let's work out the problem without an inertial couple.

$h =$

Apply the 4 criteria:



Finally, use an Energy Method:

3.3 Dynamics of a 4-Bar Mechanism

Find all pin reactions and the torque applied to crank, r_2. Gravity is in the z direction.

Data from a Kinematic Analysis:

$r_1 =$	$\theta_2 =$	$\dot{\theta}_2 =$	$\ddot{\theta}_2 =$
$r_2 =$	$\theta_3 =$	$\dot{\theta}_3 =$	$\ddot{\theta}_3 =$
$r_3 =$	$\theta_4 =$	$\dot{\theta}_4 =$	$\ddot{\theta}_4 =$
$r_4 =$			

Data for Inertial Properties:

$\overline{O_2G_2} =$	$\alpha_2 =$	$m_2 =$	$I_{G2} =$
$\overline{AG_3} =$	$\alpha_3 =$	$m_3 =$	$I_{G3} =$
$\overline{O_4G_4} =$	$\alpha_4 =$	$m_4 =$	$I_{G4} =$

3.3.1 Velocity Analysis (ME3212)

3.3.2 Free Body Diagrams

Force and Moment Balance:

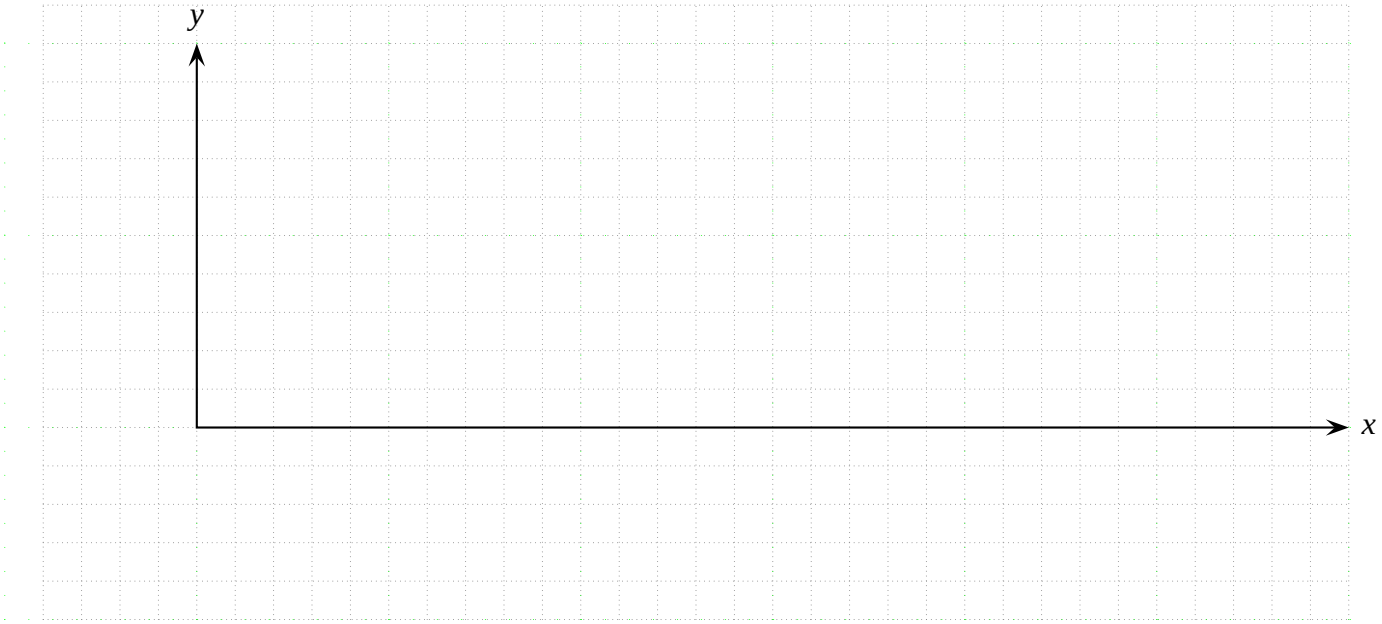
$$\sum F_x =$$

$$\sum F_y =$$

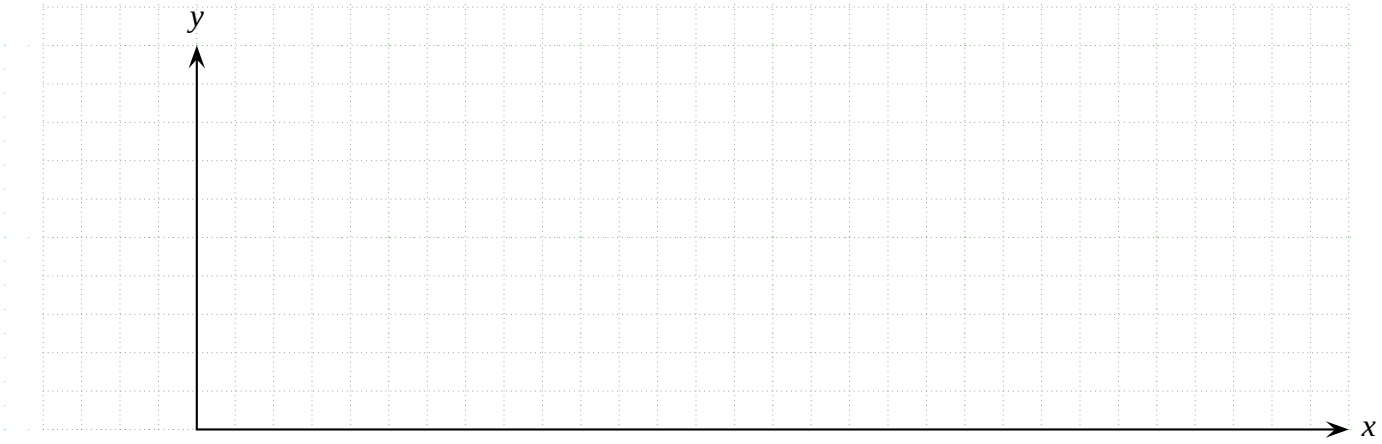
$$\sum M_{O_2} =$$

Include internal forces by developing a free body diagram for each link:

Links 2 and 4:



Link 3:



3.3.3 Equations of Motion

Link 2:

$$\sum F_x =$$

$$\sum F_y =$$

$$\sum M_{O_2} =$$

Link 3:

$$\sum F_x =$$

$$\sum F_y =$$

$$\sum M_A =$$

Link 4:

$$\sum F_x =$$

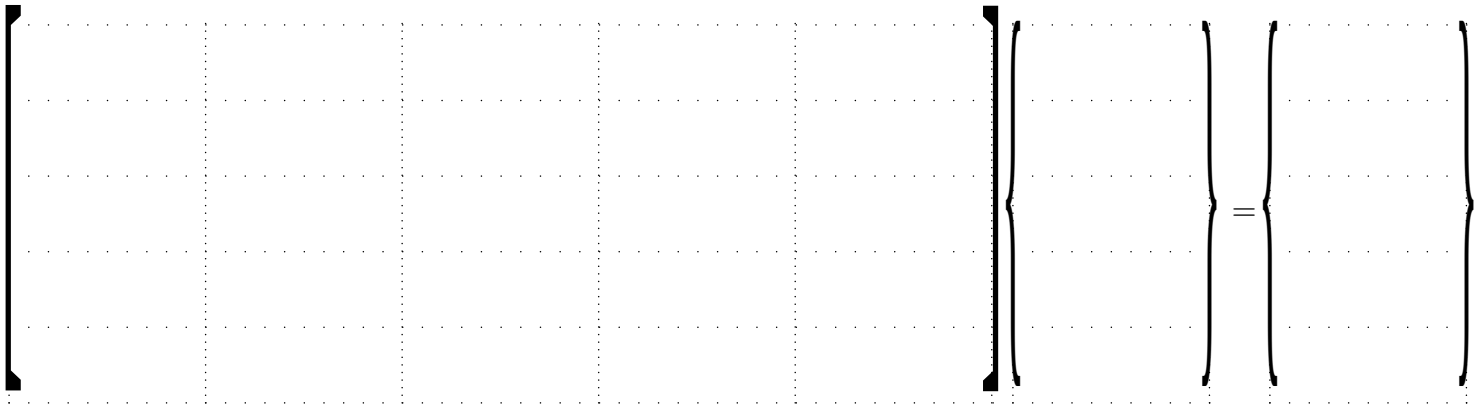
$$\sum F_y =$$

$$\sum M_{O_4} =$$

Unknowns:

3.3.4 Solution Technique

Put the coupled equations in matrix form:

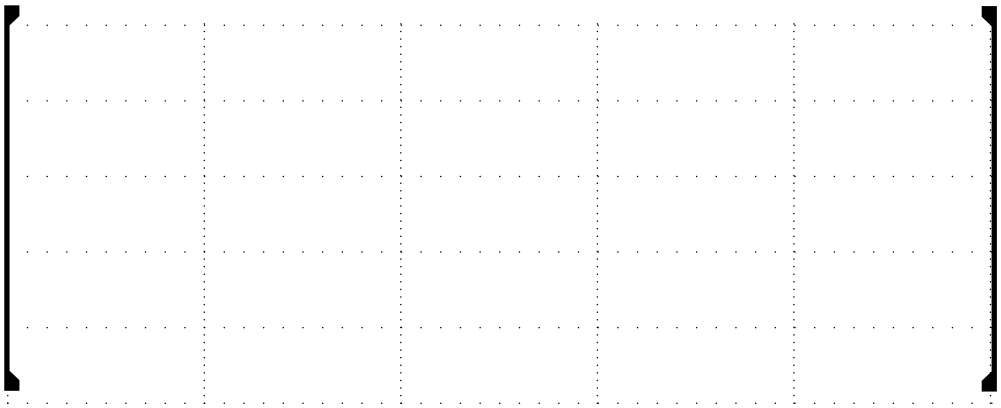


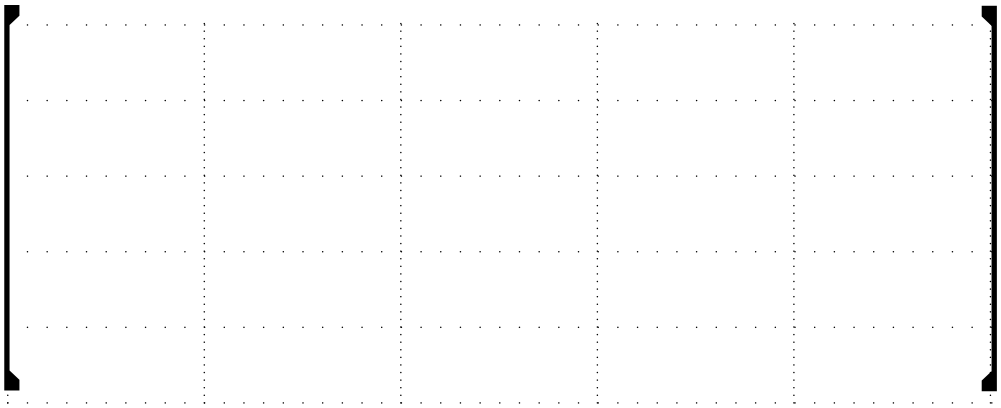
where

$$Inertia_3 =$$

$$Inertia_4 =$$

Substitute numbers:

$A =$ 

$A =$ 

$Inertia_3 =$

$Inertia_4 =$

$m_3 a_{3x} =$

$m_3 a_{3y} =$

So the inertia vector is

$$C = \left\{ \quad \right\}$$

Solve the linear system of equations:

$$A^{-1} \cdot C = \left\{ \quad \right\} = \left\{ \quad \right\}$$

Complete the solution:

$R_{2x} =$

$R_{2y} =$

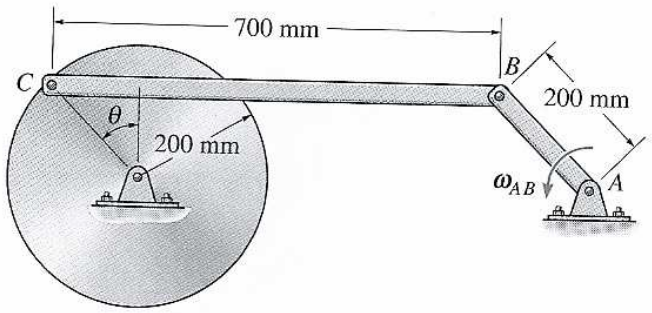
$R_{4x} =$

$R_{4y} =$

3.4 Homework Problem Set 8

1. In the following problem, gravity is acting downward.

R2-7. The uniform connecting rod BC has a mass of 3 kg and is pin-connected at its end points. Determine the vertical forces which the pins exert on the ends B and C of the rod at the instant (a) $\theta = 0°$, and (b) $\theta = 90°$. The crank AB is turning with a constant angular velocity $\omega_{AB} = 5$ rad/s.



Prob. R2–7

2. Assume the following mechanism is mounted sideways and is not influenced by gravity.

15.5 For the angular velocity of crank 2 given in the figure, find the reactions at each joint and the external torque to be applied to the crank.

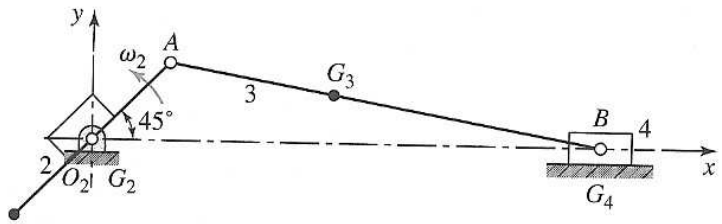
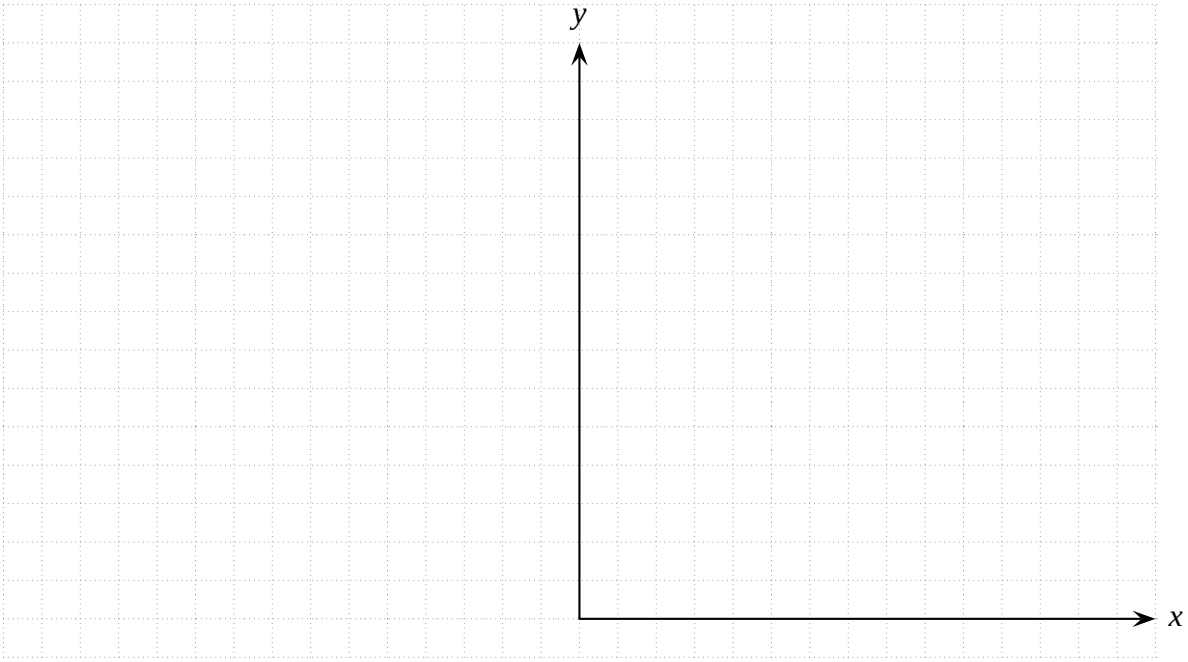


Figure P15.5 $R_{AO_2} = 3$ in, $R_{BA} = 12$ in, $R_{G_3A} = 4.5$ in, $w_3 = 3.40$ lb, $w_4 = 2.86$ lb, $I_{G_2} = 0.352$ in·lb·s^2, $I_{G_3} = 0.108$ in·lb·s^2, $\boldsymbol{\omega}_2 = 210\hat{\mathbf{k}}$ rad/s, $\boldsymbol{\alpha}_2 = \mathbf{0}$, $\boldsymbol{\alpha}_3 = -7\,670\hat{\mathbf{k}}$ rad/s^2, $\mathbf{A}_{G_3} = -7\,820\hat{\mathbf{i}} - 4\,876\hat{\mathbf{j}}$ ft/s^2, $\mathbf{A}_{G_4} = -7\,850\hat{\mathbf{i}}$ ft/s^2.

Verify your torque answer using the Lagrange Equations.

3.5 Dynamics of an Inverted Slider-Crank Mechanism

Goal: Determine the input torque and pin reactions



Given Data:

$r_1 =$	$r_2 =$	$\overline{O_3B} =$	$\overline{BC} =$
$\overline{O_2G_2} =$	$\alpha_2 =$	$m_2 =$	$I_{G2} =$
$\overline{O_3G_3} =$	$\alpha_3 =$	$m_3 =$	$I_{G3} =$
$\theta_2 =$	$\dot{\theta}_2 =$	$m_4 =$	$I_{G4} =$
$P =$			

3.5.1 Kinematic Analysis

Loop Closure Equations:

x)

y)

Square both equations and add to eliminate θ_3:

$r_3^2 =$

Expand:

$r_3^2 =$

Which gives the law of Cosines:

$r_3 =$

Similarly, the law of sines gives:

$\theta_3 =$

To determine velocities, differentiate the loop closure equations with respect to time.

x)

y)

Cast in matrix form and solve:

$$\begin{bmatrix} & \\ & \end{bmatrix} \begin{Bmatrix} \\ \end{Bmatrix} = \begin{Bmatrix} \\ \end{Bmatrix}$$

With numbers:

$$\begin{bmatrix} & \\ & \end{bmatrix}\begin{Bmatrix} \\ \end{Bmatrix} = \begin{Bmatrix} \\ \end{Bmatrix}$$

$\dot{r}_3 =$

$\dot{\theta}_3 =$

To determine accelerations, differentiate once more with respect to time:

x)

y)

Note the Coriolis term:

Write in matrix form:

$$\begin{bmatrix} & \\ & \end{bmatrix}\begin{Bmatrix} \\ \end{Bmatrix} = \begin{Bmatrix} \\ \end{Bmatrix}$$

With numbers:

$$\begin{bmatrix} & \\ & \end{bmatrix}\begin{Bmatrix} \\ \end{Bmatrix} = \begin{Bmatrix} \\ \end{Bmatrix}$$

Given the kinematic results from the loop closure equations, we can determine the accelerations of the mass centers for link 2.

Now determine the accelerations of the mass centers for link 3.

Once the complete kinematic results are know, we can set-up and solve the kinetic problem.

Free body diagram of Link 2:

Equations:

$$\sum F_x =$$

$$\sum F_y =$$

$$\sum M_{O_2} =$$

Free body diagram of Link 3:

Equations:

$$\sum F_x =$$

$$\sum F_y =$$

$$\sum M_{O_3} =$$

Free body diagram of Link 4:

Equations:

$$\sum F_x =$$

$$\sum F_y =$$

$$\sum M_{O_3} =$$

Number of Equations:

Number of Unknowns:

The last equations come from slider friction. In tangential and normal coordinates:

$F_t =$

$F_x =$

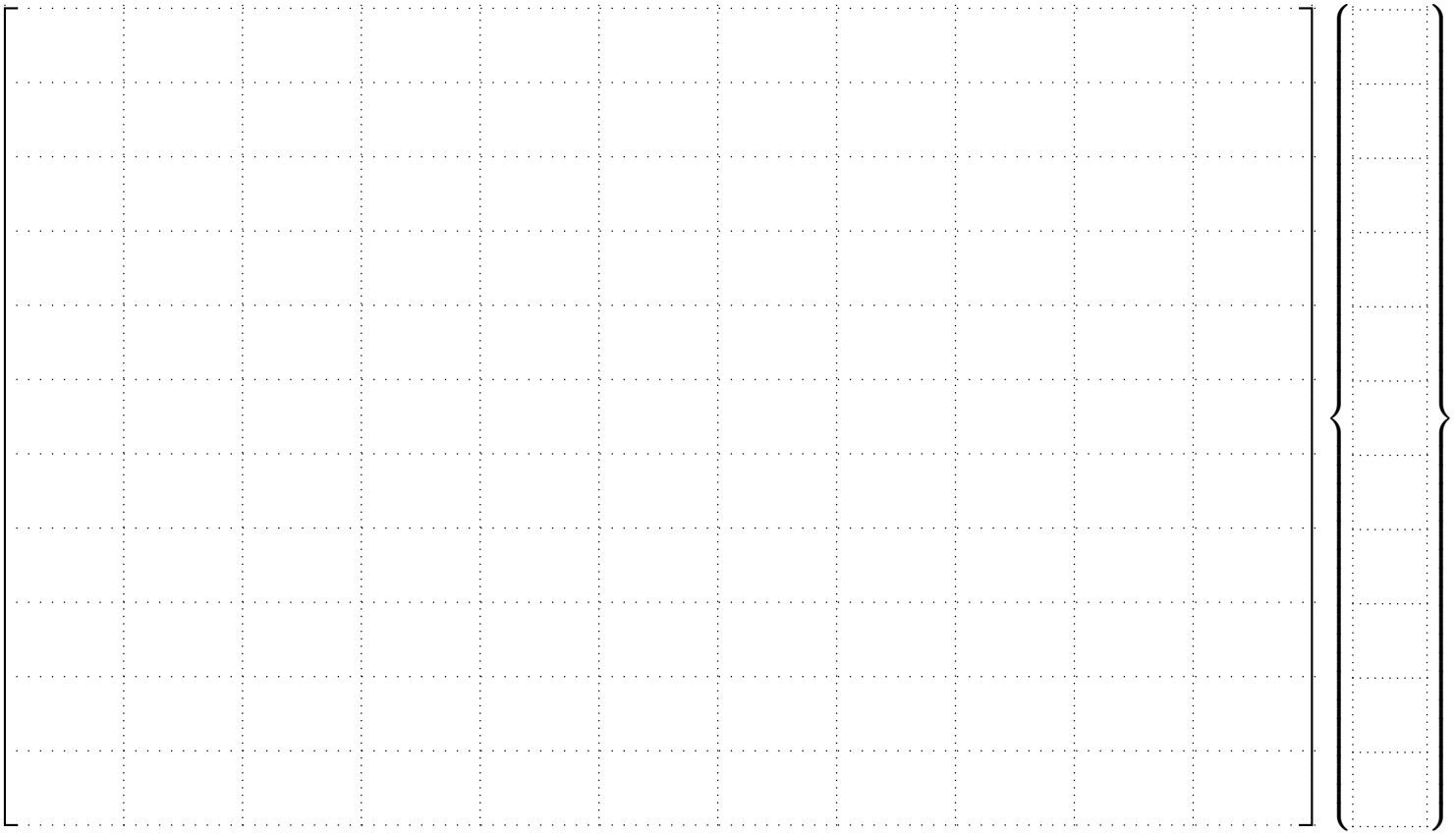
$F_y =$

If frictionless, then...

Set up the system of equations:

$$[A]\{f\} = \{B\}$$

where



$$B = \left\{ \right.$$

With Numbers:

$$B = \left\{ \quad \right\} \qquad f = \left\{ \quad \right\}$$

Thought Questions:

- What is the effect of including a larger flywheel to the crank?
- How does friction influence the solution?
- Why are the forces applied to the slider not equal and opposite?
- What is the effect of including gravity?
- How does this apply if the slider is a linear actuator?

A Mathematica notebook also accompanies this example. It can be downloaded from:

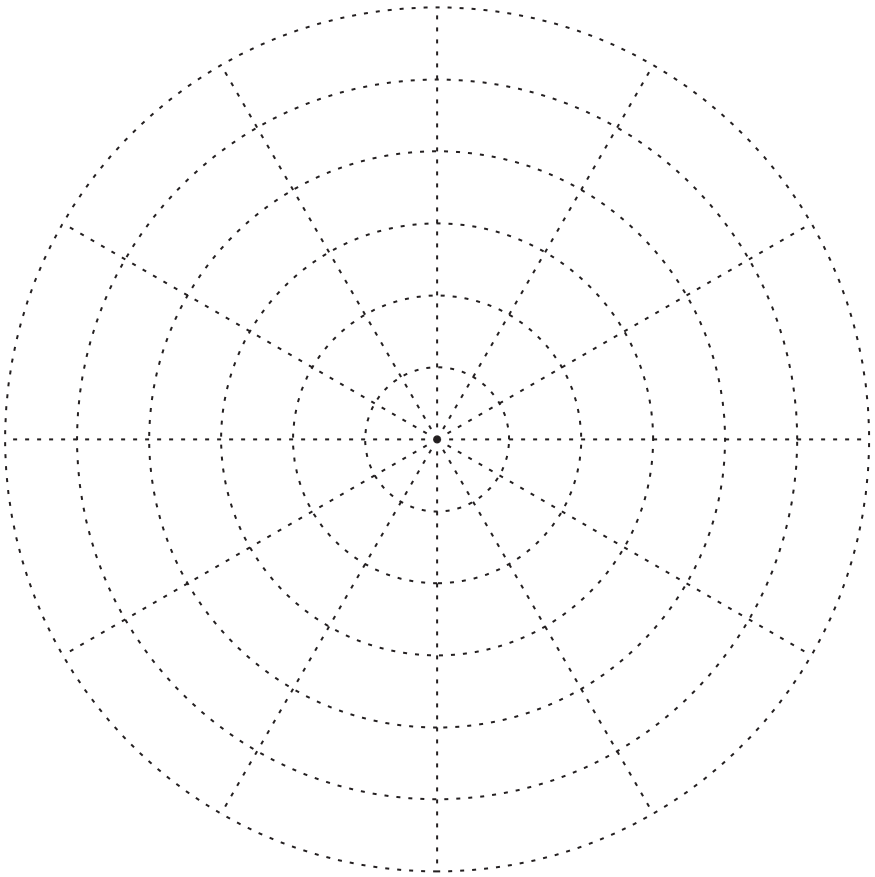
http://personal.utulsa.edu/∼jeremy-daily/ME4024/InvertedSliderCrankDynamics.nb

The PDF version of the notebook can be downloaded from

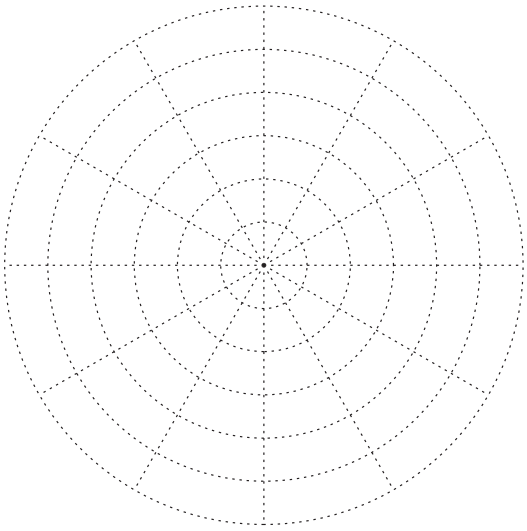
http://personal.utulsa.edu/∼jeremy-daily/ME4024/InvertedSliderCrankDynamics.pdf

4 Balancing

4.1 Balancing Rotating Masses



Balance Inertial Forces



4.1.1 Graphical Solution

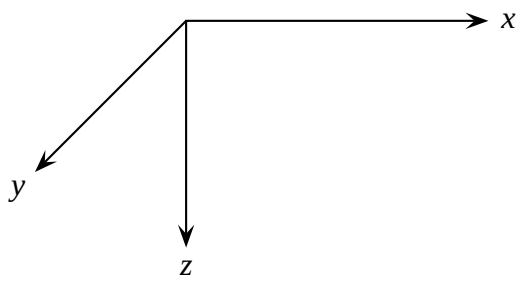
Draw a Force Polygon:

4.1.2 Analytical Solution

Set up a table:

Number	Weight	Radius	Angle	$w_i r_i \sin\theta_i$	$w_i r_i \sin\theta_i$
1					
2					
3					
Total					
Balance					

4.1.3 Static Balance

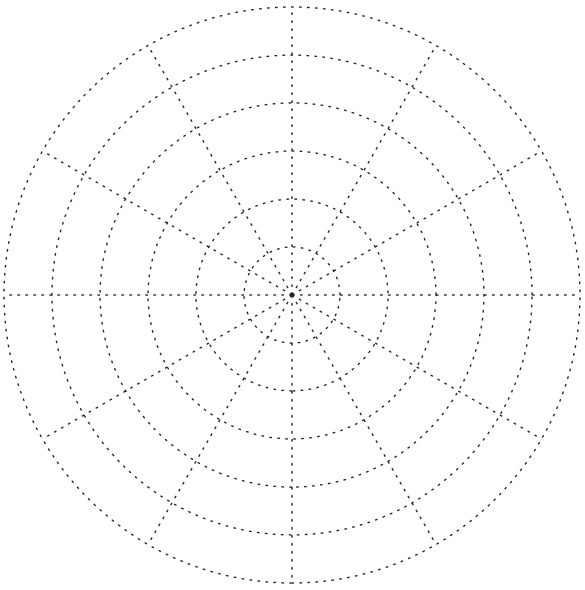


$$\sum M_{y-axis} =$$

$$\sum M_{x-axis} =$$

What along balancing along the shaft?

Example:



Build a table for moments:

Moments about A					Around Y	Around X
Number	w_i	r_i	a_i	θ_i	$w_i r_i a_i \cos\theta_i$	$w_i r_i a_i \sin\theta_i$
1						
2						
3						
				Total		
B						
				Balance		

Solve for magnitude and angle:

Force Balance				X	Y
Number	w_i	r_i	θ_i	$w_i r_i \cos\theta_i$	$w_i r_i \sin\theta_i$
1					
2					
3					
B					
			Total		
A					
			Balance		

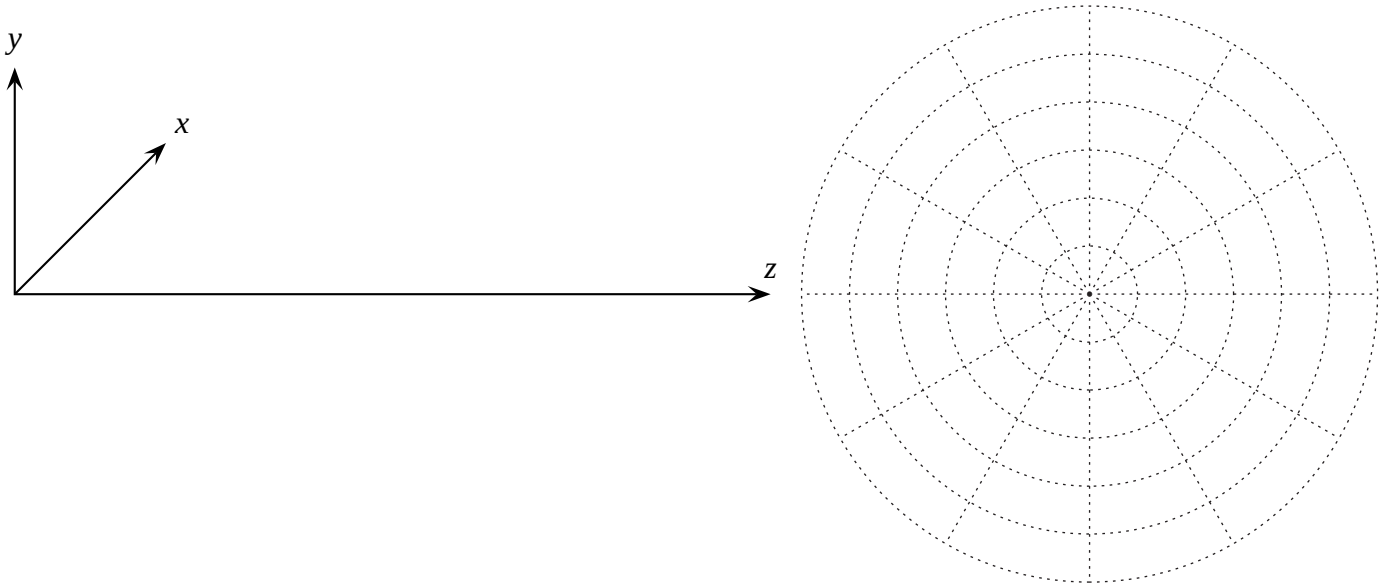
Solve for magnitude and angle:

Why is balancing important?

4.2 Dynamic Balancing

4.3 Homework Problem Set 9

1. A rigid rotor is to be balanced by the addition of a fourth mass at a 178-mm radius. Three masses already exist and they are as follows: $M_1 = 1.81$ kg, $r_1 = 381$ mm, $\theta_1 = 120°$, $M_2 = 2.27$ kg, $r_2 = 254$ mm, $\theta_2 = 225°$, $M_3 = 0.907$ kg, $r_3 = 305$ mm, $\theta_3 = 330°$. Determine the mass and angular position of the balancing mass using both graphical and analytical methods. Draw a scale diagram of the rotor using solid lines for the known masses and dashed lines for the balancing mass.
2. Determine the amounts and angular positions of two masses which, if added at a 51-mm radius in planes L and R, will balance the rotor.



3. Determine the bearing reaction forces for the unbalanced rotor in the previous problem when the rotor is spinning at 1000 rpm. The shaft is mounted vertically so gravity is not in effect.

4.4 Field Balancing

4.5 Balancing Reciprocating Masses

4.5.1 Single Cylinder Engines

4.5.2 Multi Cylinder In-line Engines

4.6 Homework Problem Set 10

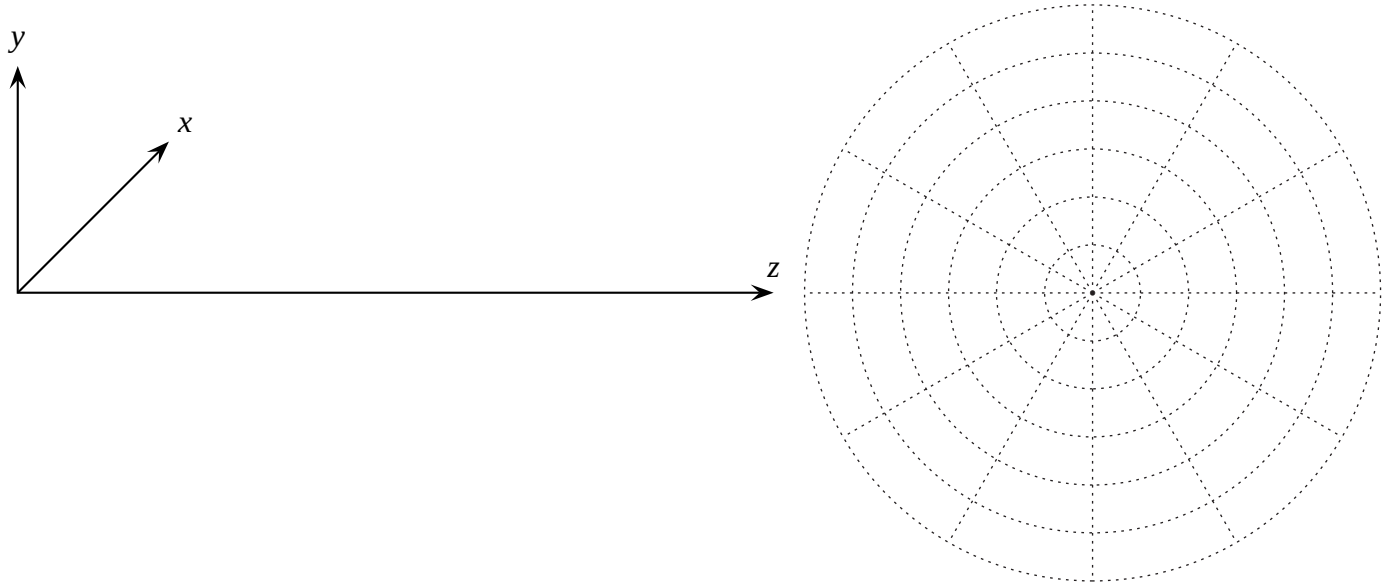
1. The following data are given for a single cylinder internal-combustion engine with the piston translating in the horizontal direction: speed = 1500 rpm, stroke = 204 mm, mass of crank and crankpin = 3.63 kg, mass of piston and piston pin = 3.18 kg, distance from crankshaft axis to the center of mass of the crankshaft = 63.5 mm, length of the connecting rod = 408 mm, mass of the connecting rod = 3.63 kg, distance from the center of mass of the connecting rod to the crank pin = 102 mm.

   a) Determine the magnitude and direction of the shaking force for a crank angle of $\theta = 150^\circ$ if no counterbalance is used.

   b) Determine the magnitude and direction of the shaking force for a crank angle of $\theta = 150^\circ$ if a counterbalance force equal to the crank inertia plus 60% of the maximum primary inertia force is used.

   c) Write a program to plot magnitude of the shaking force for the above two scenarios.

2. A proposed engine configuration is shown below. The cylinders are equally spaced with a distance *a* between their centerlines.



   a) Construct an engine balancing table and determine if any components are unbalanced.

   b) Write an expression for the shaking force in terms of θ_1 (the rotation of the first cylinder bank).

c) Write an expression for the shaking moment in terms of θ_1.

d) Write an expression for the location of the shaking force in terms of θ_1.

3. Show that an in-line 6 cylinder engine with is completely balanced in terms of primary and secondary forces and moments. The crank angles are as follows: 1 & 6 are at 0°, 3 & 4 are at 120°, 2 & 5 are at 240°. Construct a balancing table to show this.

Part II

Reference Material

5 Standards for Measurement

5.1 Measurement

Crash reconstructionists are interested in measuring things that happened in the past. Since the final results of a reconstruction are measurements of some kind, we need to understand the definitions and standards for measurements.

If we say a skid mark is measured to be 5 what does that mean? The number 5 does not mean anything unless some units are used because, for example, 5 inches is much different than 5 miles.

Developing standards for measurements was no easy task. Measurements in past times were very often based on the dimensions of human or animal body parts. For example, kings of old were so self-centered that they made lengths and masses of their own bodies and possessions the standard for the whole land. The length of the king's foot became his kingdom's standard unit of measure for the foot. Or, the inch was the width of his thumb. Kings could be replaced and so could the unit of measure. Furthermore, neighboring kingdoms had different standards. These practices, of course, made scientific progress difficult. As scientific inquiry, commerce, and colonization grew, the world began to adopt standards for measurement. Finally, the units of measure were standardized and they are still used today.

5.2 Physical Quantities and Units of Measure

The laws of physics are expressed in terms of what are known as *physical quantities*. Examples of physical quantities are force, time, velocity, temperature, and electric current. Physical quantities are categorized into *fundamental quantities* and *derived quantities*. Fundamental quantities are those quantities that are based on known standards. They are not defined in terms of other physical quantities. Derived quantities are those quantities that are based on other physical quanti-

Prefix	Abbreviation	Exponent	Multiplier
Tera	T	10^{12}	1,000,000,000,000
Giga	G	10^{9}	1,000,000,000
Mega	M	10^{6}	1,000,000
Kilo	k	10^{3}	1000
Hecta	H	10^{2}	100
Deca	D	10^{1}	10
deci	d	10^{-1}	0.1
centi	c	10^{-2}	0.01
milli	m	10^{-3}	0.001
micro	μ or u	10^{-6}	0.000 001
nano	n	10^{-9}	0.000 000 001
pico	p	10^{-12}	0.000 000 000 001
fempto	f	10^{-15}	0.000 000 000 000 001

Table 5.1: Definition of prefixes for factor of ten relationships

ties. The units of measure associated with fundamental quantities and derived quantities are called *fundamental units* and *derived units*, respectively.

There are two major systems of measurement used in the world today. The first, used mainly in America, is the US* system. The second is the SI system (metric system), which has been adopted by the rest of the world, including Canada. SI is the abbreviation for *International System of Units* (from the French *Le Système International d'Unités*). Both systems are used throughout this text, with the majority of the examples worked in the US system.

The metric system uses fundamental units that are related by factors of ten.† This is convenient because only the fundamental units need to be known. For example, if we know the length of a meter, then a kilometer is 1000 times the length of a meter. Table 5.1 lists the different names that are prepended to the fundamental unit in order to change it. Some of these will be familiar from our experience with consumer electronics. Outside the US the metric system has been adopted

*Sometimes referred to as the English or Imperial system.
†A factor of ten is also known as an order of magnitude.

by most countries and the scientific community for their technical publications. However, typical Americans (juries) can relate much easier to the US system. This book presents concepts using both systems of measurement.

5.2.1 Fundamental Units

There are seven fundamental units, also known as *base units*, that define the SI system. The seven base units are:

1. Length
2. Mass
3. Time
4. Electric Current
5. Temperature
6. Amount of Substance
7. Intensity of Light

The definitions of the seven base units are established through the General Conference on Weights and Measures. The first three, length, mass, and time, are familiar to crash investigators and are discussed below. The last four are not used as frequently in the crash investigation world; therefore, the definitions are presented without discussion.

5.2.1.1 Standard of Length

SI

The *meter* (m) is the fundamental unit of length measure in the metric system. The definition of a meter as reported in Ref. [1] is “The length of the path travelled by light in a vacuum during a time interval of 1/299,792,458 of a second.” An earlier definition was based on 1/40,000,000 of Earth’s polar circumference. This standard may seem arbitrary, but it defines the length of a meter in terms of the speed of light, which is a universal constant. This allows for a consistent, accurate, and accessible reproduction of the standard meter anywhere in the universe.

All other measures of length in the metric system are determined by the prefixes and the multipliers listed in Table 5.1. Therefore, if we know the measure of a length in meters, then the measure

in centimeters is determined by moving the decimal two places to the left. This is the same as multiplying by 0.01 or dividing by 100.

US

The fundamental unit of measure for length in the US system is the *foot*. Other common measures of length include the mil, inch, yard, and mile. Less common measures are the palm, rod, furlong, fathom, and league. Table 5.2 on page 5-6 shows the relationships between the common units of measure for length.

5.2.1.2 Standard of Mass

SI

The *kilogram* (kg) is the fundamental unit of measure in the metric system. The definition of the kilogram as quoted from Ref. [1] is "The kilogram is equal to the mass of the international prototype of the kilogram." The international prototype is a platinum cylinder kept at the International Bureau of Weights and Measures near Paris, France. Obviously, the kilogram is 1000 grams, but more interestingly, one liter of pure water weighs one kilogram. The liter of water that has a mass of 1 kg will weigh approximately 2.2 lb. As a matter of fact, the mass of a liter of water used to be the definition of a kilogram.

US

Mass in the US system is measured in *slugs*. A slug is the mass of an object that will accelerate at the rate of 1 ft/s^2 given the application of 1 lb of force. Thus, an equivalent unit for a slug is:

$$1\,\text{slug} = \frac{1\,\text{lb}}{1\,\frac{\text{ft}}{\text{sec}^2}} = 1\,\frac{\text{lb-sec}^2}{\text{ft}} \tag{5.1}$$

Most physical phenomena under interest occur at the earth's surface, and the weight of an object is defined by the acceleration due to gravity. Therefore, in order to retain the proportions in Eq. (5.1), a slug must weigh 32.2 lb on the earth's surface. By similar argument, we can determine the mass of any object on the surface of the earth if we know its weight:

$$m = \frac{w}{g} \tag{5.2}$$

where m is in slugs or $\frac{\text{lb-sec}^2}{\text{ft}}$,
w is in pounds, and
$g = 32.2\,\frac{\text{ft}}{\text{sec}^2}$, which is the acceleration due to gravity on the earth's surface.

Considerations

Mass is a measure of the amount of matter an object contains. Objects with a higher mass tend to weigh more when they are in similar environments. Mass is not affected by the environment where it is observed. For example, a very large log has the same mass both in and out of the water; however, it may take a large machine to move the log on land, but a boy could maneuver the log while it is in the water. Another example would be taking a cannon ball to the moon and being able to lift it quite easily. Its mass never changes because it still has the same amount of matter, however, its weight changes due to the lower gravitational pull on the moon. The relationship between mass and weight is defined by Newton's Second Law in Section 10.2 on page 10-2.

When we weigh something in the US system, we actually are measuring its force due to the gravitational pull on the earth. Because the pull of gravity is constant, the mass can be determined using Eq. (5.2). Sometimes you may see the mass reported as *lbm*, which means pound-mass. This means the mass is the same as an object that weighs one pound on the earth.

When we weigh something in the SI system a spring scale measures force but reports the object's mass in kilograms. This scale would be no good on the moon. Because the spring scale is really measuring a force, the force on the moon would be smaller than that on the earth. However, since the pull of gravity on the earth is constant, the conversion of weight to mass is valid.

There are scales that do not convert force into mass, and they retain their accuracy regardless of astronomical location. These scales are called *balances*, which use levers and linkages to balance a known mass with an unknown mass. Older powder scales and the stand-up scales at the doctor's office are examples of balancing scales.

The bottom line here is that when using the US system, scales measure force (weight) and report force. However, scales in the metric system also measure force but report mass.

5.2.1.3 Standard of Time

The *second* is the unit of measure for time. This is common to both the US and SI systems of measurement. The definition of the second from Ref. [1] is "The duration of 9,192,631,770 periods

TIME

1	minute	=	60	seconds			
1	hour	=	60	minutes	=	3600	seconds
1	second	=	0.01667	minutes	=	0.0002778	hours

LENGTH

1	inch	=	25.4	millimeters	=	2.54	centimeters
1	foot	=	12	inches	=	0.3048	meters
1	mile	=	5280	feet	=	1.609	kilometers
1	mil	=	0.001	inches	=	0.0254	mm
1	millimeter	=	0.03937	inches			
1	centimeter	=	0.3937	inches			
1	meter	=	39.37	inches	=	3.28	feet
1	kilometer	=	0.621	miles	=	3280	feet

MASS

1	lbm	=	0.4536	kilograms	=	0.031	lbf-sec^2/ ft
1	slug	=	32.2	lbm	=	14.6	kilograms
1	kilogram	=	2.20	lbm	=	0.0685	slugs
1	lbf-sec^2/ft	=	1	slug	=	0.016	tons (mass)

FORCE

1	lbf	=	4.45	newtons	=	0.031	poundals
1	newton	=	0.225	lbf	=	0.102	kgf

PRESSURE

1	atmosphere	=	14.7	psi	=	101.3	kPa
1	psi	=	6.9	kPa	=	0.068	atm

ANGLES

1	degree	=	0.01745	radians	=	60	minutes
1	radian	=	57.29	degrees			
1	revolution	=	360	degrees	=	2π	radians

of the radiation corresponding to the transition between two hyperfine levels of the ground state of the cesium-133 atom (at a temperature of 0 K).” An earlier definition was 1/86,400 of a mean solar day in the year 1900.

Other measures of time include minutes, hours, days, and weeks. What about months? The reason months are not used as a basis for measurement is because the number of seconds will change depending on which month is being used; for example, February has a different amount of time than January. The definition of a year is one complete revolution of the earth around the sun. This revolution occurs in 365.25 days, thus a leap year is required every 4 years. With the advent of scientific breakthroughs such as radio communication and quantum theory, a more consistent time standard was needed. This need led to the development of atomic clocks, which can keep time to within 1 second over 5000 years. Table 5.2 shows how many seconds are in the standard measures of time.

Crash investigators use seconds and hours as the main measures of time, where 1 hour is 3600 seconds. Often, crashes happen in a blink of an eye and the time measure is usually in milliseconds. Crashes often last around 100 milliseconds, which is 0.1 seconds. The second can be prefixed with the terms listed in Table 5.1 on page 5-2.

Example 5.1. How many microseconds in a minute?

There are 60 seconds in a minute and $10^6 = 1,000,000$ microseconds in a second, which gives 60,000,000 microseconds per minute.

5.2.1.4 Standard of Electric Current

The *ampere* (A) is the fundamental unit of measure of electric current in both the US and SI systems of measurements. The ampere is defined as “That constant of current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed one meter apart in a vacuum, would produce between those conductors a force of $2x10^{-7}$newton per meter of length.” [1]

5.2.1.5 Standard of Temperature

The *kelvin* (K) is the fundamental unit of measure of temperature in the SI system. The definition of kelvin is: “The fraction of 1/273.16 of the thermodynamic temperature of the triple point of water.” [1] Other temperature scales include Celsius in the SI system, and Fahrenheit and Rankine

in the US system. The Kelvin and Rankine scales have a zero point at absolute zero, which is the temperature at which molecular motion ceases. The common Celsius and Fahrenheit scales are related with the following formula:

$$F = \frac{9}{5}C + 32$$

5.2.1.6 Standard of Amount of Substance

The *mole* (mol) is the fundamental unit of measure of amount of substance. A mole is defined as: "The amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilograms of carbon 12. When the mole is used, the elementary entities must be specified and may be atoms, molecules, ions, electrons, other particles, or specified groups of such particles." [1]

5.2.1.7 Standard of Intensity of Light

The *candela* (cd) is the fundamental unit of measure for the intensity of light. The definition of candela is: "The luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency $540x10^{12}$ hertz and that has a radiant intensity in that direction of 1/683 watt per steradian." [1]

5.2.2 Derived Units

A derived unit is a quantity that can be written in terms of the fundamental quantities of time, length, and mass. There are other fundamental measurements in physics that have to do with electricity and light, but traffic crash reconstructionists rarely need these.

5.2.2.1 Velocity and Speed

These quantities are most easily understood in terms of miles per hour (kilometers per hour) because this is what we use everyday when driving. The difference between velocity and speed is that velocity is used in the context of both magnitude and direction. Speed is purely magnitude. A more detailed discussion of this concept is found in Section 7.2 on page 7-3.

Velocity and speed are usually represented as some length divided by time. Length and time are fundamental quantities.

$$\text{Units of velocity} = \frac{\text{Units of length}}{\text{Units of time}} \tag{5.3}$$

5.2.2.2 Acceleration

Acceleration is the time rate of change of velocity. Another way of looking at it is to consider how much the velocity changed over a given amount of time, by dividing the units of velocity by time.

$$\text{Units of acceleration} = \frac{\text{Units of velocity}}{\text{Units of time}} \tag{5.4}$$

$$\text{acceleration} = \frac{\frac{\text{length}}{\text{time}}}{\text{time}} = \frac{\text{length}}{\text{time}^2}$$

The most common acceleration value is the acceleration of an object due to free fall on the earth:

US	SI
$32.2 \frac{\text{feet}}{\text{second}^2}$	$9.81 \frac{\text{meters}}{\text{second}^2}$

Notice both quantities are written in terms of length per time^2, which are both fundamental quantities. The units can be read as either “feet per second squared” or “feet per second per second” because they mean the exact same thing.

5.2.2.3 Force

Force derives its units from Newton’s Second Law, $F = ma$, described in Section 10.2 on page 10-2. Mass is a fundamental quantity and we saw from Section 5.2.2.2 that acceleration is also described by fundamental units. This means that force can be written as a combination of fundamental quan-

tities. We will always start with $F = ma$:

$$\text{Units of force} = \text{Units of mass} \times \text{Units of acceleration} \tag{5.5}$$

US

$$1\,\text{unit of force} = 1\,\text{slug} \times \frac{1\,\text{ft}}{\text{second}^2}$$
$$1\,\text{unit of force} = \frac{1\,\text{slug} \cdot \text{ft}}{\text{second}^2}$$

This is the definition of the *pound*. The abbreviation for pounds is lb but when talking strictly about force, the abbreviation, lbf (pound-force), is often used to distinguish from lbm (pound-mass). As an example, someone's weight could be correctly reported as "200 slug·feet per second squared", which means "200 pounds". Remember, the center dot (·), the short dash (-), and the times symbol ($\times$) all mean multiplication when dealing with units.

As an aside, the weight of one slug is called a *poundal* and is equal to 32.2 lb.

SI

By a similar argument,

$$1\,\text{unit of force} = 1\,\text{kilogram} \times \frac{1\,\text{meter}}{\text{second}^2}$$
$$1\,\text{unit of force} = \frac{1\,\text{kg} \cdot \text{m}}{\text{second}^2}$$

Instead of constantly writing or saying "the force pushed with 200 kilogram meter per second squared," the unit called a *newton* is used to quantify the amount of force in the metric system. There are 4.448 newtons in one pound. An apple weighs about 1 newton.

Force in the metric system has an equivalent measure in terms of the kilogram-force (kgf), which is the weight of one kilogram on the earth. This means there are 9.81 newtons to a kgf.

5.2.2.4 Angles

Angle measures are the rotational analogue of length. The units of angle measures are common to both systems. The different angles are best illustrated by the circle in Figure 5.1. The most common system is the degree system, in which there are 360° in a circle.

Each degree can be divided by 60 to get minutes. These angle measures are found in ballistic studies concerning the accuracy of firearms. The trajectory of the bullet may only vary a few minutes of angle. The advantage of reporting precision this way is it is independent of the range of the target.

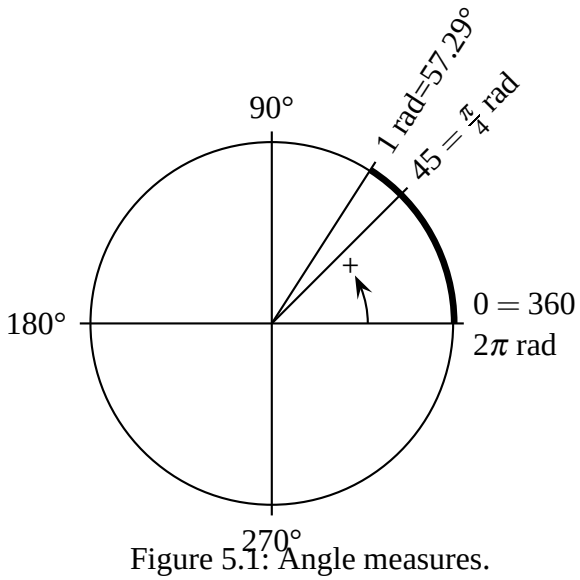


Figure 5.1: Angle measures.

Furthermore, each minute of angle can be divided once more into seconds of angle. These measurements are used in surveying equipment, like a total station. The angle and distance precision of modern surveying equipment must be very good to resolve small details at long ranges.

Radian measure is used in mathematics because it relates the radius of a circle to the arc length. One radian is the angle swept out by laying one radius on the circumference of a circle. The dark line on the circle of Figure 5.1 on page 5-11 is exactly one radius long, thus defining one radian. There are 2π radians in a full circle and π radians in a half circle. Most calculators have the constant π built in as 3.1415926... The value of π is an infinitely long, non-repeating decimal that has never been fully determined. For the purposes of reconstructing traffic crashes, the value of π can be used as 3.14.

Revolutions are another way of measuring angles. There is simply 1 revolution in a circle. A quarter revolution is 90°, and so forth. Revolutions are mainly used in determining the rotational velocity of something, as in revolutions per minute (rpm).

5.2.2.5 Pressure

Pressure is a measure of the force on a given area. Often, pressure is associated with air or water, but pressure also exists in solids. One unit of pressure that does not belong to either system is the atmosphere. An *atmosphere* is the amount of air pressure at sea level.

$$\text{Units of pressure} = \frac{\text{Units of force}}{\text{Units of area}} \tag{5.6}$$

US

Any unit of length can be converted into an area by squaring it. In this sense, the square inch is the most common unit of measuring area, followed by the square foot when dealing with traffic crashes. That said, the most common unit for pressure in the US system is the pound per square inch, or psi. The fundamental units of pressure measured in psi can be written as

$$1\,\text{psi} = \frac{1\,\text{lb}}{1\text{in}^2} = \frac{1\,\text{slug} \times \frac{1\,\text{foot}}{\text{second}^2}}{1\,\text{in}^2} = \frac{1\,\text{slug} \times \frac{12\,\text{in}}{\text{second}^2}}{1\,\text{in}^2} = \frac{12\,\text{slugs}}{\text{in} \cdot \text{sec}^2}$$

The other common form of reporting pressure is through the use of a barometer. This gives the pressure in terms of inches of mercury (in. Hg) or inches of water. For crash reconstruction, these units are never used.

SI

The unit of area is the square meter and the unit of force is the newton, so

$$1\,\text{unit of pressure} = \frac{1\,\text{newton}}{1\,\text{meter}^2} = \frac{1\,\frac{\text{kg}\cdot\text{m}}{\text{second}^2}}{1\,\text{meter}^2} = \frac{1\,\text{kg}}{1\,\text{m}\cdot\text{s}^2}$$

This unit of pressure is named the *Pascal* and abbreviated as Pa. Since the newton weighs about as much as an apple and a square meter is about half the size of a twin bed, one Pascal measures very small pressure. Therefore, values are often reported as kiloPascals (kPa), megaPascals (MPa), and gigaPascals (GPa). There are 103.3 kPa at 1 atmosphere and 6.895 kPa in 1 psi.

The strength of materials is given in terms of the *stress*, which is the force per unit area.

5.2.2.6 Density

Density is simply a measure of the amount of mass per unit volume. Since volume is a length cubed, the fundamental units are inherently present. The common measures of density are the pound mass per cubic inch in the US system and the kilogram per cubic meter in the metric system. Also, the metric system works such that water has a density of 1 gram per cubic centimeter (1 g/cc).

$$\text{Units of density} = \frac{\text{Units of mass}}{\text{Units of volume}} \tag{5.7}$$

Density can be interpreted in a number of different ways. A term like “population density” is a quantity counting the number of people per area (not volume). Context is important when using the term “density”.

5.2.2.7 Work and Energy

There are many units used to describe work and energy but the fundamental units are the same. Some examples of energy units include:

- British Thermal Unit (BTU)
- Joule
- erg
- calorie
- ft-lb
- kilowatt-hour
- horsepower-hour
- electron volt

We will only consider the ft-lb for the US system and the Joule for the metric system for the purposes of reconstructing traffic crashes. The fundamental units of work and energy come from the definition of work, which is the product of force and displacement (see Section 13.1 on page 13-1).

$$\text{Units of work} = \text{Units of force} \times \text{Units of length} \tag{5.8}$$

US

The ft-lb is an obvious result of the definition of work. To arrive at the fundamental units, substitute the fundamental units of force from Section 5.2.2.3 into Eq. (5.8):

$$1\,\text{ft-lb} = \frac{1\,\text{slug}\cdot\text{ft}}{\text{second}^2} \times 1\,\text{ft} = \frac{1\,\text{slug}\cdot\text{ft}^2}{\text{sec}^2}$$

SI

Similarly for the metric system,

$$1\,\text{unit of work} = \frac{1\,\text{kilogram}\cdot\text{meter}}{\text{second}^2} \times 1\,\text{meter} = \frac{1\,\text{kg}\cdot\text{m}^2}{\text{sec}^2}$$

where the unit of work in this case is called the *Joule*. Careful examination of the fundamental units for work show that they are the product of the units for mass and the square of velocity.

$$\text{Units of work} = \text{Units of mass} \times (\text{Units of velocity})^2$$

5.2.2.8 Torque and Moments

A torque is the same thing as a moment, and is the rotational equivalent of force. Torque is generated by a force acting on a lever arm to create a twist. Since torque is the product of force and length, the units are the same as work and energy. However, the two phenomena are very different. First, torques and moments act in a certain direction, that is, they are vector quantities. Energy and work have no direction. Second, torque is not conserved, whereas energy is always conserved.

The units for torque are commonly inch·pounds or foot·pounds in the US system. The units for torque in the metric system are newton-meters or newton-millimeters.

5.2.2.9 Power

Power is the time rate at which work is done, $P = \frac{W}{t}$, so

$$\text{Units of power} = \frac{\text{Units of work}}{\text{Units of time}} \tag{5.9}$$

Rearrangement of Eq. (5.9) will give units of work. This is why some measures of work and energy have power and time measurements.

US

By far, the most common unit for power in the United States is the horsepower (hp). Originally, one horsepower was the ability of a horse to lift water (do work) from an English mine shaft in a certain amount of time. Eventually, the unit was standardized as 550 ft·lb/sec. By recalling the fundamental units of work from Section 5.2.2.7, we know that the fundamental units of power in the US system are

$$1\,\text{Unit of power} = \frac{\frac{1\,\text{slug}\cdot\text{ft}^2}{\text{sec}^2}}{\text{sec}} = \frac{1\,\text{slug}\cdot\text{ft}^2}{\text{sec}^3}$$

where this unit is shortened slightly to the ft·lb/sec. It is apparent how the unit hp·hour is a unit of energy or work based on the rearrangement of Eq. (5.9).

SI

The ironic twist in naming units is present in the standard unit of power for the metric system named after James Watt. He suggested the concept of the horsepower and as a result, they named the metric unit a Watt. A Watt is defined in terms of fundamental units as

$$1\,\text{Watt} = \frac{1\,\text{J}}{\text{sec}} = \frac{\frac{1\,\text{kg}\cdot\text{m}^2}{\text{sec}^2}}{\text{sec}} = \frac{1\,\text{kg}\cdot\text{m}^2}{\text{sec}^3}$$

Domestic energy bills are determined based on the kilowatt·hour. If one kilowatt·hour of energy is consumed over a time span of one hour, then the power consumption is one kilowatt.

Further analysis of the dimensions of power reveal that the product of force and velocity gives the fundamental units of power. Therefore, it takes more power to lift something quickly than to lift it slowly– even though the same amount of work is being done.

For example, consider a professional basketball player weighing 200 lb who jumps 36 inches. He does the same amount of work as a 200-lb man who climbs a 36-inch step ladder because they move a force, their weight, a certain distance. The basketball player, however, is more powerful because the time it took him to get to 3 feet is shorter than the step ladder climber.

5.2.3 Factors and Ratios

There are two common ratios that are used in traffic crash reconstruction. The first is the acceleration factor and the second is the coefficient of friction.

The acceleration factor is defined by the actual acceleration rate divided by the acceleration rate due to gravity:

$$f = \frac{a}{g}$$

where g is either 32.2 ft/s/s or 9.81 m/s/s. The distinct advantage of reporting an acceleration factor is that there are no units and the number means the same in both systems of measurement. Even though there are no fundamental units associated with an acceleration factor, a label of G is usually applied to call attention to the fact that it is an acceleration factor that is referenced to gravity.

The coefficient of friction is a measure of the ratio of two forces: the weight of an object and the force required to drag the object across a particular horizontal surface. This is defined with the Greek letter μ as

$$\mu = \frac{F_T}{w}$$

Factors and ratios of quantities provide a convenient way to represent physical quantities while retaining sufficient information to be useful in all systems of measurement.

5.2.4 Dimensional Analysis

Dimensional analysis is a very powerful tool to check your work and to understand the physics of a problem. Dimensional analysis transcends the traffic crash reconstruction field and can be useful in many of the physical sciences. Basically, the mathematical operations that are performed on the numbers are also performed on the units. Every physical quantity that has a value must also have

a unit of measure. There are rules when working with physical quantities. If a ratio or factor is desired, then the units must cancel to give a dimensionless result.

All units can be broken into the fundamental units through dimensional analysis. However, it is often much more convenient to make names for commonly used units that involve more than one fundamental unit.

Here are some rules for combining physical quantities:

1. Only like dimensional quantities can be added, subtracted, compared, or equated.

   5 cars + 3 wheels = 8 ???? This does not make any sense. The units of an addition or subtraction problem must agree. If they do not, then a conversion factor must be applied so both quantities have a common unit.

2. Numbers that are multiplied or divided must also have their units multiplied or divided. This is the hallmark of unit conversions. Continuing with the wheels and cars example, we would have to convert 5 cars to 20 wheels.

3. Quantities that are raised to a power must also have their units raised to a power.

Example 5.2. What is 5 feet less than 30 meters? Give your answers in inches and centimeters.

The problem can be written mathematically as

$$30\,\text{meters} - 5\,\text{feet} = ?$$

Convert all units to inches by using the conversion factors listed in Table 5.2 on page 5-6:

$$5\,\text{feet} \times \frac{12\,\text{inches}}{1\,\text{foot}} = 60\,\text{inches}$$

$$30\,\text{meters} \times \frac{39.37\,\text{inches}}{1\,\text{meter}} = 1181.1\,\text{inches}$$

The problem can be solved because the units are now common for the subtraction:

$$1181.1\,\text{inches} - 60\,\text{inches} = 1121.1\,\text{inches}$$

To convert to centimeters, Apply another conversion factor from Table 5.2. If a direct conversion does not exist, then multiple steps may be required. Since there are 100 centimeters per meter

(Table 5.1 on page 5-2),

$$1121.1\,\cancel{\text{inches}} \times \frac{1\,\cancel{\text{meter}}}{39.37\,\cancel{\text{inches}}} \times \frac{100\,\text{centimeters}}{1\,\cancel{\text{meter}}} = 2847.6\,\text{centimeters}$$

Notice how the meters and inches canceled in the last example. This method of canceling units is the core of unit conversion.

Dimensional analysis goes beyond conversions because it typically combines units of different physical quantities. Take, for example, speed or velocity for which the result is going to be some unit of length divided by some unit of time. If the result of a calculation and its corresponding dimensional analysis does not result in some length divided by time, then a mistake has been made somewhere in the calculation. Dimensional analysis provides confidence in calculated answers because the numbers have physical meaning. Most of the derivations in this book rely on the concepts of dimensional analysis.

5.3 Accuracy and Precision

When we are called upon to investigate or reconstruct a traffic crash, we often need to take various measurements. We assume that the measurements are valid and truly represent the length of the skid mark, the distance of the chord, the drag factor of the road, or whatever else we are measuring.

Relative Difference between two numbers is calculated by dividing the difference between two numbers by their average:

$$\text{Percent Difference} = \frac{a-b}{\left(\frac{a+b}{2}\right)} \times 100 \tag{5.10}$$

Example 5.3. What is the relative difference between 0.73 and 0.69?

Apply Eq. (5.10) with $a = 0.73$ and $b = 0.69$:

$$\begin{aligned}\text{Percent Difference} &= \frac{0.73-0.69}{\left(\frac{0.73+0.69}{2}\right)} \times 100 \\ &= 5.63\%\end{aligned}$$

If we have a list of numbers, we can determine a relative distance between the high and low values, and all other numbers in the list will be within that relative difference.

Precision is the relative agreement between several measurements that have been made in the same manner. When we read a tape measure in order to measure the length of an unknown, and then repeat the measurement and compare the two measurements, we are checking the precision of our measurement. We may calculate the relative deviation of our measurements by the use of a simple equation:

$$P_r = \frac{M_c - M_a}{M_a} \tag{5.11}$$

where P_r = Relative Deviation
M_c = the current measurement
M_a = the average measurement

Example 5.4. Let us say that we measure a single skid mark four times and we get the following values: 102 ft, 102.5 ft, 101 ft, and 102.5 ft. Let us then measure the mark a fifth time. This time, we measure the mark at 103 ft in length. What is the relative deviation of this fifth measurement?

First we must calculate an average:

$$M_a = \frac{102 + 102.5 + 101 + 102.5 + 103}{5} = 102.2$$

Next we apply Eq. 5.11:

$$P_r = \frac{103 - 102.2}{102.2} = 0.00783$$

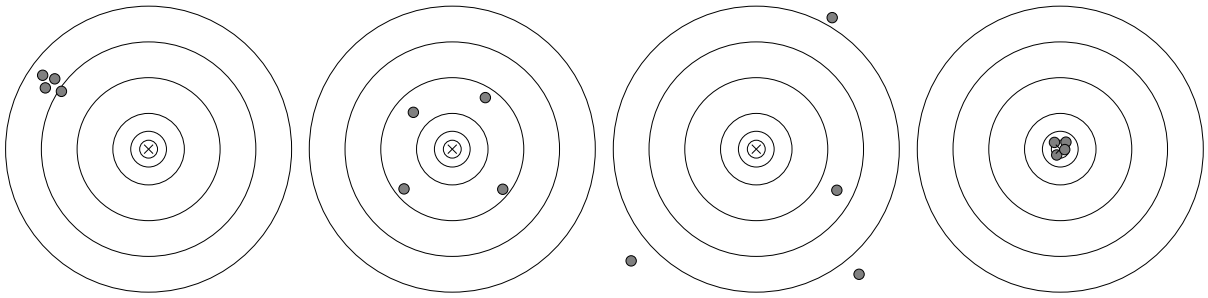
We find that P_r has a value of 0.783%.

Accuracy of a measurement describes how close a given measurement comes to a known physical standard. We may calculate the relative error of a measurement in a similar manner to that outlined for the calculation of relative deviation:

$$R_e = \frac{M_c - M_t}{M_t} \tag{5.12}$$

where R_e = Relative Error
M_c = current measurement
M_t = true measurement

Example 5.5. Let us measure a known distance of 100 feet with two different measuring devices. The first device measures the 100-foot distance as 99 feet. The second device measures the distance as 102 feet. What is the relative error of the two devices?



(a) Precise (b) Accurate (c) Neither accurate or precise (d) Both precise and accurate

Figure 5.2: Targets illustrating the concepts of accuracy and precision.

In this example, the first device has a relative error of -0.01, or -1%. The second device has a relative error of 0.02, or 2%. Hence, we might conclude that the first device is the one to use, as it is more accurate. However, we would want to test the devices several times and also to calculate the relative deviation of each instrument. We may find that one device is significantly more precise than the other, which may then make it the instrument of choice.

In Figure 5.2 we will illustrate the difference between accuracy and precision by means of an example that will resonate with those of us who are, or have been, sworn law enforcement officers. Every so often, we are required to qualify with the handguns that we carry. Let us say that one officer shoots a very small group, but that this group is in the shoulder of the target. We say that his or her shooting is precise but not accurate. Another officer centers his or her shot group, but the group is the size of a large dinner plate. That officer's shooting is accurate but not precise. A third officer shoots a group the size of the plate and partially off the target. His or her shooting is neither accurate nor precise. The sergeant then shoots the 10 ring out of the target. We say that the sergeant's shooting is both accurate and precise. This is the type of measurement that we wish to take at our crash scenes.

5.4 Numerical Precision

The techniques of traffic crash reconstruction predate the development of digital computers and calculators. As such, the calculations have typically been performed by hand. When doing hand calculations, a trade-off exists between the time to obtain a result and the amount of precision the result should have. From that trade-off, calculations in traffic crash reconstruction have traditionally been done with two numbers to the right of the decimal point. This is not a hard and fast rule, rather

a tradition coming from the economics of time and the significance of the scene measurements.

When we use a number to represent some physical quantity, that quantity must be measured. Measurements are never exact representations of the actual quantity. If we have a precise and accurate measurement of some quantity, then the measurement is considered to be good. The precision of a measurement is represented by the number of *significant figures* and the accuracy is determined through comparison to a known standard.

5.4.1 Significant Figures

Significant figures are the digits that are meaningful to the actual measurement. These digits are counted from the left of the number. Trailing zeros are not counted unless a decimal point is included or the zero is on the right of a decimal point. Here are some examples:

Number	Significant Figures	Number	Significant Figures
327	3	3270.	4
3.27	3	327.00	5
3270	3	3.270×10^{-3}	4
330	2	3.30	3

When arithmetically combining numbers, judgement plays the most important role in determining the significant figures of the result. However, there are a few accepted conventions based on the significant figures of the input. Those conventions are:

- When adding or subtracting numbers, the result should have no more significant figures than the least precise number. For example:

  $$2.98 + 30.2091 \approx 33.2$$

  Obviously, our calculator will tell us something different, but the difference is not significant. If the value of 2.98 came from an accurate physical measurement, then the actual value of the quantity could be anywhere from 2.975 to 2.984. Because of this, the lesser digits of the above addition are both uncertain and insignificant.

- When multiplying or dividing measurement values, the result should not contain more significant figures than the least precise number. For example:

  $$\frac{5600}{32.2} \approx 170$$

  The calculator gives 173.913043478. Since 5600 only has two significant figures, then the result should also have two significant figures. However, if the precision of the numerator is increased to 5 significant figures, then the denominator would limit the result to 3 significant figures. For example:

  $$\frac{5605.1}{32.2} \approx 174$$

  In this case, the calculator gives 174.071428571.

- When calculating values from trigonometric functions, at least three significant figures should be used.

- The tradition in traffic crash reconstruction is to carry between 2 and 5 significant figures for all hand calculations and intermediate steps. This translates to a simple rule of keeping two decimal places for all intermediate calculations (except 3 decimal places for trigonometric functions).

The conventions of significant figures are usually ignored when using a computer or a graphics calculator that is capable of easily storing highly precise numbers. However, the extra precision of the computer does *not* give better, more accurate results. The precision of the final result should not exceed the precision of the input values.

5.4.2 Rounding

Since the input measurements to some calculations have a limited number of significant figures and the calculator always exceeds the necessary numerical precision, how do we figure out what number to report?

One technique is to *round* the result for the correct number of significant digits. Rounding rules are as follows:

- Keep all significant figures unchanged and drop all trailing figures if the first of the dropped figures is a 4 or less.

- Increase the least significant digit by 1 and drop all trailing figures if the first of the dropped figures is a 5 or greater.

Here are some examples of rounding:

Original Number	Significant Figures	Rounded Number
327	2	330
0.70710678	3	0.707
32.17	3	32.2
29.938	2	30.
254.2752	3	254

It is important to realize that rounding can only be performed once. In other words, it is correct to round 2.145 to 2.15 but 2.15 cannot be continued to be rounded to 2.2. As you can see, if we are to round 2.145 to two significant figures, the result would be 2.1.

5.4.3 Truncating

Truncating is the second method (rounding is the first) of achieving the correct number of significant figures. Truncating is simply the process of dropping all the insignificant figures of a number. Below is an example of truncating numbers:

Original Number	Significant Figures	Truncated Number
327	2	320
0.70710678	3	0.707
32.17	3	32.1
29.938	2	29
254.2752	3	254

We recommend truncating all final speed answers to the nearest whole number. For all other calculations, we must either round or truncate to the correct number of significant figures.

5.5 Quantifying Uncertainty

The laws of physics and the math used to reconstruct traffic crashes is precise, that is, the formulas give explicit results. The advent of computers and calculators lets people compute values to many decimal places. The precision of these machines is wonderful, but our interpretation of this must be tempered by the possible variations of the input variables.

All quantities that can be measured for a traffic crash reconstruction have some measurement tolerance. For example, a tape measure can give values down to the 1/16”, so any carefully measured item would have about 1/16” tolerance associated with it. However, many items needing measurement in reconstruction are not well defined. Two examples are the drag factor and skid mark distances. Rarely will two drag factor tests give the same values into the third digit, and determining the beginning of a skid mark is subjective in nature. A detailed study can be found in Ref. [2].

A reconstructionist can never know the exact way a crash transpired. Therefore, the best we can do is to provide a range of possible values. All of the calculations are performed using high and low values, so the final answer will have a high and a low value. The other alternative is to always choose values as to give someone the benefit of the doubt. There are other ways of quantifying and propagating uncertainty that are beyond the scope of this text that include Monte Carlo simulations, sensitivity analysis, and error propagation techniques.

Exercises

Exercise 5.1. A tire is inflated to 35 psi. What is the pressure in kPa? 241

Exercise 5.2. A road slope is measured with a four foot level with one end 15 cm off the ground. What is the slope of the road

a.) in percent? 12.3%

b.) in degrees? 7.01°

c.) in radians? 0.122

Exercise 5.3. A vehicle has a mass of 1500 kg. What is its weight on the surface of the earth

a.) in pounds? 3300

b.) in newtons? 14715

c.) What is its mass in slugs? 102.73

Exercise 5.4. An equation for the friction coefficient is $\mu = \dfrac{F}{w}$. Let $F = 100$ newtons and $w = 100$ pounds. What is the value of μ? 0.22

Exercise 5.5. Show that 550 ft-lb/sec $=$ 746 watts.

Exercise 5.6. Show that 32.2 ft/sec^2 $=$ 9.81 m/sec^2.

Exercise 5.7. What is the relative precision of the third skid test when

$f_1 = 0.65$

$f_2 = 0.72$

$f_3 = 0.70$

1.4%

Exercise 5.8. A surveyor sets two stakes exactly 62 feet apart.

12.9% a.) What is the error of measurement when the distance measured by pacing is 70 ft?

1.1% b.) What is the error of measurement when the distance measured by a tape measure is 62.7 ft?

-1.4% c.) What is the error of measurement when the distance measured by a roller wheel is 61.1 ft?

Exercise 5.9. Determine the value of $\cos 30^\circ$

0.866 a.) to three significant figures.

0.8660 b.) to four significant figures.

6 Vectors

6.1 Vector and Scalar Quantities

An important concept in traffic crash reconstruction is the difference between a scalar quantity and a vector quantity. The effect of this difference as it relates to various scalar and vector quantities is detailed in this chapter. To begin, let's look at a couple examples before we give a definition of vectors and scalars.

Consider the instruction given to a police officer to stop a vehicle traveling at a speed of 35 mph away from the scene of a crash, or the instruction given to a worker to place a picture hanger on a wall 5 inches from a defined point on that wall. Does either instruction provide enough information for the person to take action?*

Now consider if it were stated that the vehicle was traveling 35 mph eastbound on Elm Street or that the picture hanger should be located 5 inches directly to the right of the defined mark. With this additional information, the respective tasks can be accomplished.

The first set of instructions consists of scalar quantities, whereas the second set consists of vector quantities. Scalar quantities possess magnitude only, whereas vector quantities possess both magnitude and direction. Some common scalar quantities are time, distance, energy, temperature, mass, and density. Common vector quantities are force, velocity, momentum, displacement, and torque.

Speed in itself is also a scalar quantity, defining only the amount of distance covered with respect to time, regardless of the changes in direction that the object may undergo during its travel. Velocity, on the other hand, is a vector quantity, incorporating direction with speed. Velocity correlates with displacement, the straight-line distance that separates some initial point from which the object travels to some other point reached by the object. The object may have reached this point via a circuitous route, but its velocity will be calculated as if it had traveled the shortest distance. Distance and displacement do not necessarily have the same magnitude. They are not the same quantity.

*The answer is no because there is no direction information.

A scalar is a quantity that only has magnitude. It can be a constant or a variable. A scalar is considered one dimensional.

A vector is a multidimensional quantity, in other words, a vector has both magnitude and direction. A vector needs both components for its description.

Table 6.1: Definition of Scalars and Vectors

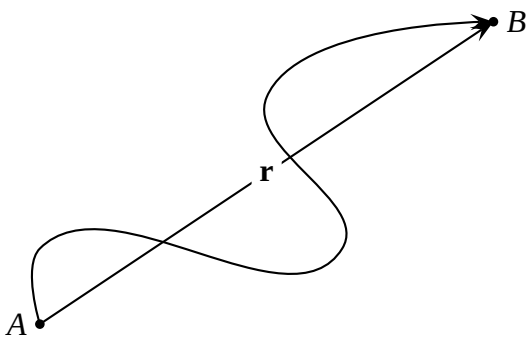


Figure 6.1: The path and displacement from point *A* to point *B*.

For example, consider a particle that travels from point *A* to point *B*, taking the curved path shown in Figure 6.1. The particle covers a certain distance as it travels along this path. In doing so, the particle experiences a certain displacement represented by the displacement vector **r**. The magnitude of the path and of the displacement are clearly not the same. The magnitude of the path is larger.

If a 400-meter sprinter runs around the track in the inside lane, he or she will travel a distance of 400 meters. However, since the start and finish are coincident, there will be no displacement.

Because scalar and vector quantities have different properties, mathematical operations using vector quantities are handled differently than when scalar quantities are used. Scalar quantities can be manipulated by the ordinary rules of algebra. Mathematical operations using vectors will be covered in following sections.

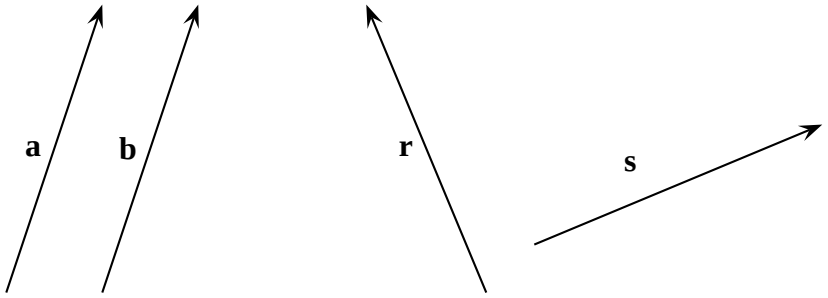


Figure 6.2: Vectors with the same magnitudes.

6.2 Vector Basics and Drawing Vectors

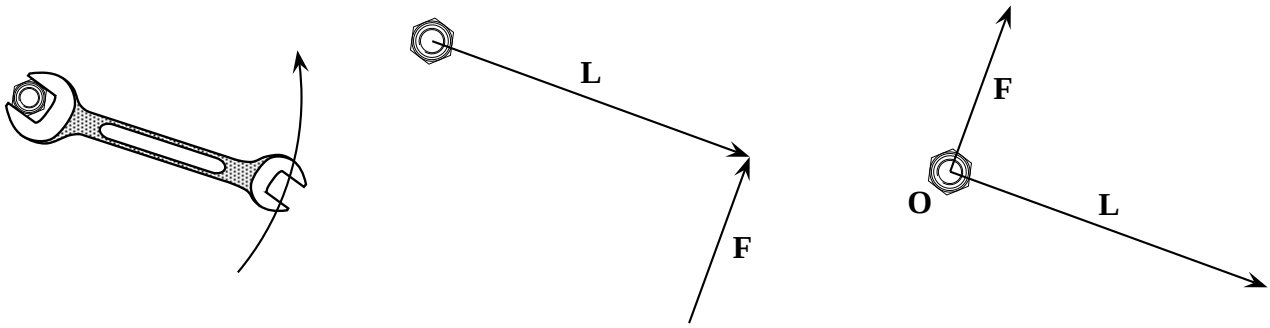
When drawing a vector quantity, the vector quantity is represented by a straight line with an arrowhead at one end. The direction of the arrow is significant because it shows the direction of the vector; the arrowhead is said to indicate the sense of the direction (e.g., whether the vector is pointing northeast rather than southwest). This direction must be strictly maintained when drawing the vector on paper or when moving the vector in a drawing. The length of the line is also significant and is drawn proportional to the magnitude of the quantity (i.e., the longer the line, the larger the magnitude). As with the direction of the vector, the length of the vector must be strictly maintained. If either the direction or length of a vector is inadvertently changed when drawing or moving it, it is no longer the same vector.

Consider Figure 6.2, in which vectors **a** and **b** are equal because their lengths, (i.e., their magnitudes), are the same, and their directions are the same, since they are parallel. Vectors **r** and **s**, however, are not equal even though their magnitudes are the same because these vectors do not have the same direction.

Symbols representing vector quantities are normally printed in boldface type: **a**, **F**, etc. Usually italic type is used for scalar quantities and for the magnitude of vectors. When vector quantities are handwritten, an arrow or a line is usually placed above the symbol: $\vec{a}$, or $\bar{a}$, or sometimes under the symbol: $\underline{a}$.

Vectors in space originate at a point commonly called **O** and end at a point called the *terminal point*. Many times when vectors are drawn together to illustrate a problem or an action, they are shown starting at points other than the origin. This is fine since such drawings most likely make what is being illustrated more understandable. However, for computational purposes, the vectors originate at **O**, as well as act at **O**.

For example, consider Figure 6.3a, in which a force **F** is being applied to a wrench of length **L** to



(a) Physical System (b) Vector Representation (c) Vectors through Point **O**

Figure 6.3: The vector representation of a wrench turning a nut.

remove the nut.

A common vector representation of this action is shown in Figure 6.3b. It is intuitive and makes the action easy to visualize. However, the correct form for mathematical analysis is shown in Figure 6.3c since in reality, the force vector **F** is acting at point **O**, the nut. This matter of the origin will be explored in greater detail in the section on cross products (Section 6.6.3).

When a vector is drawn in a coordinate system, a three-axis system is used, with the axes labeled *x*, *y*, and *z* for each of the three dimensions. In Figure 6.4, which should be viewed as a three-dimensional drawing, vector **a** connects the origin to the point at (x_1, y_1, z_1). The points are subscripted with the number 1 to denote the uniqueness of those particular coordinates.

Vector **a** thus has the coordinates (x_1, y_1, z_1) and is written:

$$\mathbf{a} = (x_1, y_1, z_1)$$

The three coordinates are also called the components of the vector. Usually, a vector will be written with its components indexed with the axis:

$$\mathbf{a} = (a_x, a_y, a_z)$$

where a_i is the magnitude of vector **a** in the direction of the i^{th} axis. Other times, the vector may be

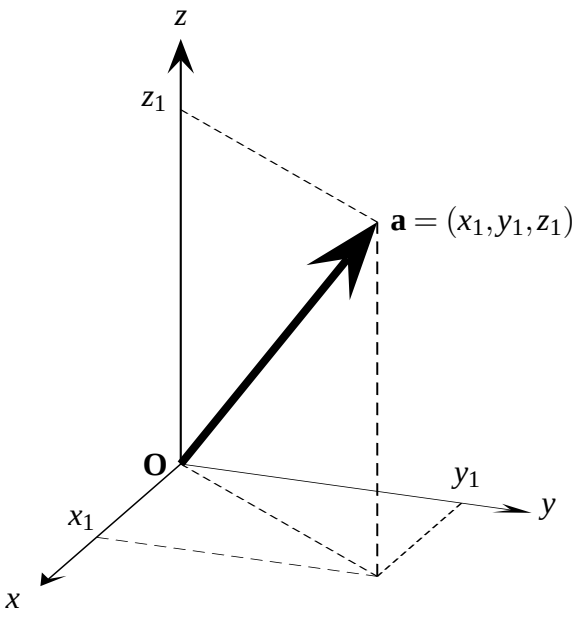


Figure 6.4: A three-dimensional representation of the vector **a**.

written in terms of numbers where, for example, 1 refers to x, 2 refers to y, and 3 refers to z:

$$\mathbf{a} = (a_{1,}a_2, a_3)$$

6.3 Length of a Vector

Given the coordinates of a vector, the length of that vector can be found in similar fashion to finding the length of a line segment.

Consider Figure 6.5, in which vector **a** is shown with coordinates (x_1, y_1, z_1). The subscript 1 indicates the first unique point under consideration. Finding the formula for the length of the vector is basically a two-step process. Consider the (x,y) plane first. Call the projection of vector **a** on the (x,y) plane "vector **r**". It can be seen that a right triangle is formed on the x-y plane, with **r** as the hypotenuse and the x_1 and y_1 components of **r** as the sides. Thus, the length of **r** can be found by the Pythagorean Theorem:

$$r^2 = x_1^2 + y_1^2 \tag{6.1}$$
$$r = \sqrt{x_1^2 + y_1^2}$$

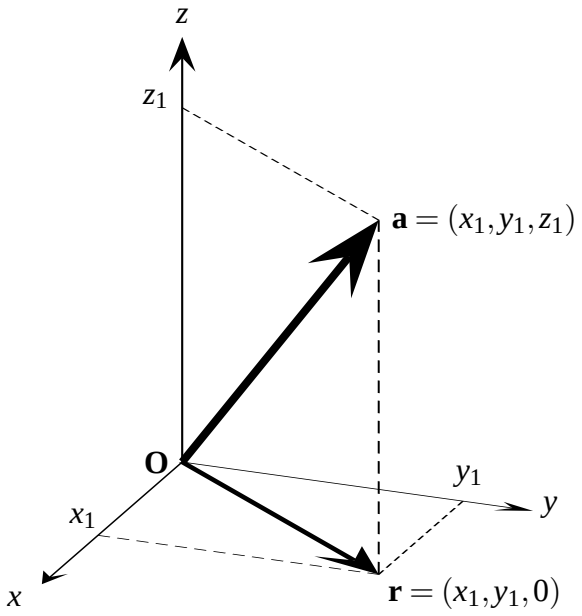


Figure 6.5: The component of the vector **a** projected on the (x,y) plane is vector **r**.

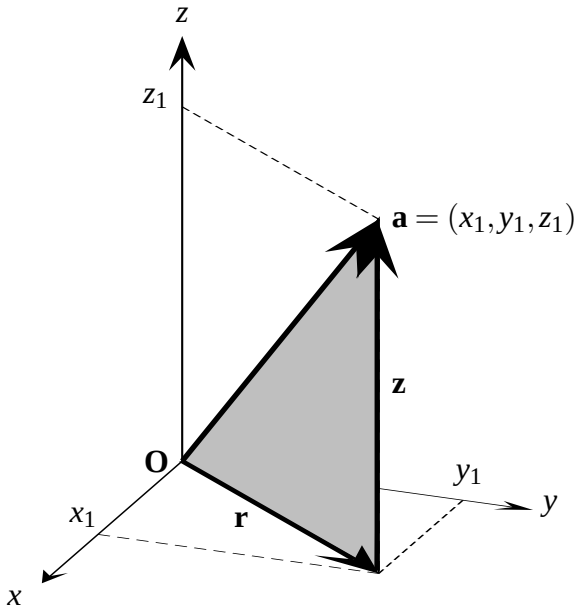


Figure 6.6: The vectors **r** and **a** form a right triangle.

Next, consider the vertical plane containing vectors **r** and **a**. It can be seen in Figure 6.6 that another right triangle is formed, with **a** as the hypotenuse, with the z component of **a** as one side, and with **r** as the other side. Thus, the length of **a** can be found by the Pythagorean Theorem:

$$a^2 = r^2 + z_1^2 \tag{6.2}$$

From Eq. (6.1) it was determined that $r^2 = x^2 + y^2$, so substituting this into Eq. (6.2) yields:

$$\begin{aligned} a^2 &= r^2 + z_1^2 \\ a^2 &= x_1^2 + y_1^2 + z_1^2 \\ a &= \sqrt{x_1^2 + y_1^2 + z_1^2} \end{aligned} \tag{6.3}$$

which is the equation to find the length of vector **a**. If we generalize the procedure for any vector (x,y,z), then the length of a vector is simply:

$$|\mathbf{a}| = a = \sqrt{x^2 + y^2 + z^2} \tag{6.4}$$

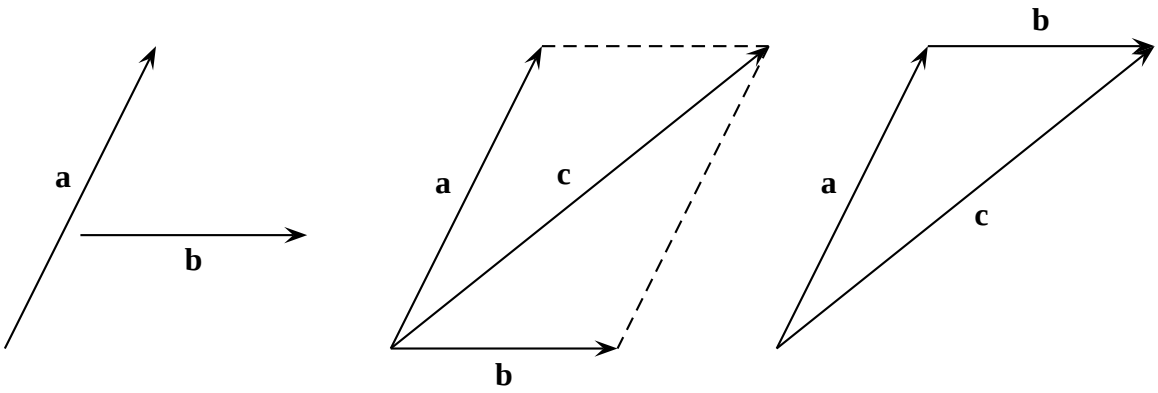
The vertical bar notation represents the norm or absolute value. This means the norm or absolute value of a vector is its length (magnitude) and is denoted by either an italic letter, a, or with vertical bars, $|\mathbf{a}|$.

6.4 Vector Addition

Adding two vectors graphically is accomplished by using the parallelogram method or by the head-to-tail method. Consider Figure 6.7 in which we wish to add vector **a** and vector **b**.

By the parallelogram method, shown in Figure 6.7b, the tails of the vectors are placed together at the origin **O**. These vectors form two sides of a parallelogram. Construct a third side from the head of vector **a** parallel to vector **b**. Then construct the fourth side from the head of vector **b** parallel to vector **a**. Now draw in vector **c** from the origin diagonally across the parallelogram to the opposite corner. Vector **c** is called the *resultant* and, being the sum of vector **a** and vector **b**, is the answer. Thus $\mathbf{c} = \mathbf{a} + \mathbf{b}$.

By the head-to-tail method, shown in Figure 6.7c, the tail of vector **b** is placed at the head of vector **a**. The resultant, vector **c**, is drawn in from the tail of vector **a** to the head of vector **c**. As before,



(a) Two vectors (b) Parallelogram method (c) Head-to-tail method

Figure 6.7: Graphically adding vectors using the parallelogram method and the head-to-tail method.

$\mathbf{c} = \mathbf{a} + \mathbf{b}$.

Vectors can also be added by adding their corresponding components:

$$(a_x, a_y, a_z) + (b_x, b_y, b_z) = (a_x + b_x, a_y + b_y, a_z + b_z)$$

6.4.1 Vector Addition Example

Add the vectors $\mathbf{a} = (2,3,4)$ and $\mathbf{b} = (-1,0,1)$.

Solution

Each component is added separately so the x component of the resultant is $2 + (-1) = 1$, the y component is $3 + 0 = 3$, and the z component is $4 + 1 = 5$. So:

$$(2,3,4) + (-1,0,1) = (1,3,5)$$

Vector addition can be easily shown on a graph. Consider Figure 6.8 on page 6-9, in which vector **a** and vector **b** are added together.

Considering only the x component in this example (since the demonstration would be the same for the y and z components), it can be seen that the sum of the projections of each vector onto the horizontal axis of the x,y plane equals the projection of the resultant vector onto that axis.

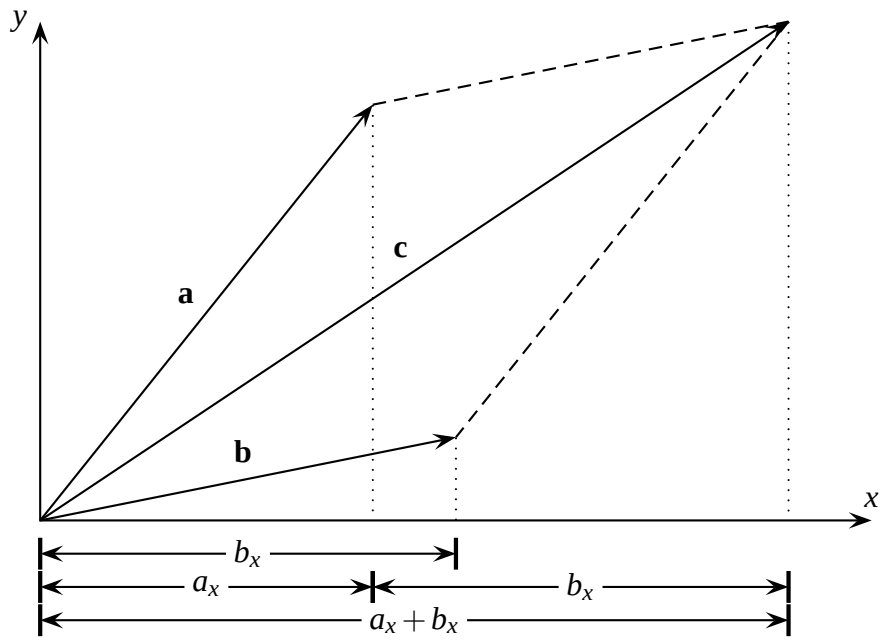


Figure 6.8: Adding the *x* components of vectors **a** and **b**.

There are two properties of scalar addition which also hold true for vector addition. One is the Commutative Property, which states that the order in which vectors are added makes no difference:

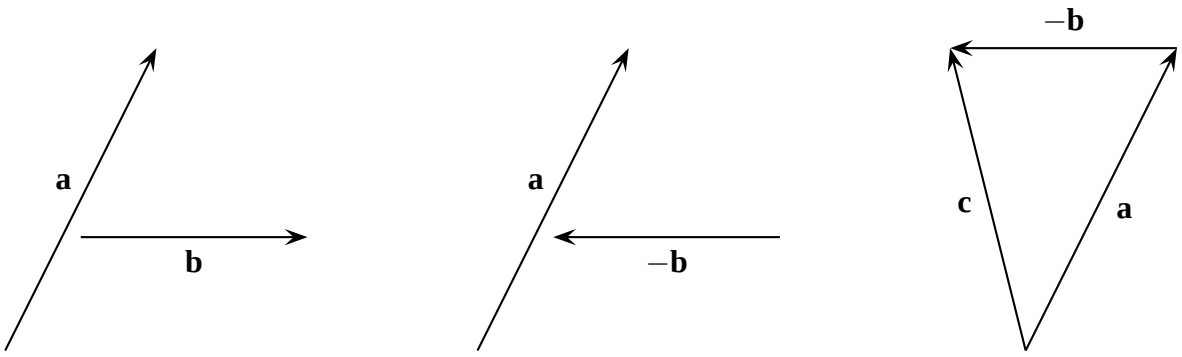
$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

> COMMUTATIVE PROPERTIES OF VECTORS
> When any two vectors are added together, the order in which they are added does not matter.

The other is the Associative Property, which states that when a number of vectors are added together, the grouping order does not matter:

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$$

> ASSOCIATIVE PROPERTY OF VECTORS
> When any number of vectors are added together, the order in which they are grouped does not matter.



(a) Two vectors

(b) Inverse of vector **b**

(c) Result is adding the inverse

Figure 6.9: An illustration of subtracting vector **b** from vector **a**.

6.5 Vector Subtraction

There are times when one vector must be subtracted from another. Consider the case in Figure 6.9a, in which vector **b** is to be subtracted from vector **a**. To do this, we can again apply a rule from scalar arithmetic (from Section **??**), which states that when subtracting, change the sign of the subtrahend and follow the rules of addition. Thus, if we wish to subtract **b** from **a**, we must find the negative of vector **b** and then add it to vector **a**:

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$$

Finding the negative of a vector means finding a vector that is equal in magnitude, i.e., the same length, but points in the opposite direction, as in Figure 6.9b. Thus, adding vector $-\mathbf{b}$ to vector **a** produces resultant vector **c**, as in Figure 6.9c.

6.6 Vector Multiplication

Only vectors of the same kind can be added together, i.e., only velocity vectors can be added to velocity vectors, only displacement vectors can be added to displacement vectors, etc. It works just like the addition of scalar quantities, in which seconds are added to seconds, pounds are added to pounds, but feet are not added to degrees. Further, just like scalars, vectors of different kinds can be multiplied together to obtain new quantities. Vectors have direction in addition to magnitude, so the conventional rules of scalar multiplication do not totally apply. Thus, three different types of

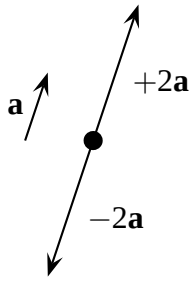


Figure 6.10: An illustration of vector **a** being multiplied by $+2$ and -2.

multiplication operations for vectors will be presented:

1. A vector multiplied by a scalar yielding a vector.
2. Vectors multiplied together yielding a scalar.
3. Vectors multiplied together yielding another vector.

6.6.1 Vector–Scalar Multiplication

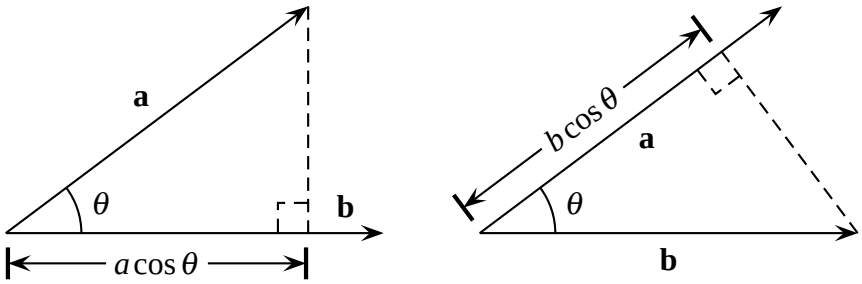
Multiplication of a vector by a scalar produces a new vector whose magnitude is simply the magnitude of the scalar quantity times the magnitude of the vector quantity. If p is the scalar quantity and **a** is the vector quantity, their product is written $p\mathbf{a}$. The new vector has the same direction as **a** if p is positive, but will have the opposite direction if p is negative. In component notation:

$$p\mathbf{a} = p(a_x, a_y, a_z) = (pa_x, pa_y, pa_z)$$

In Figure 6.10, for example, if vector **a** is multiplied by 2, a scalar quantity, then the result is a vector twice as long as vector **a** in the same direction as **a**. If vector **a** is multiplied by -2, a negative scalar quantity, then the result is a vector twice as long as **a** in the opposite direction to **a**.

6.6.2 Dot Product

Two vectors can be multiplied together with a scalar product as a result. Consider two vectors, **a** and **b**, and the angle θ between them, as shown in Figure 6.11. The scalar product of vector **a** and



(a) Projection of **a** onto **b**

(b) Projection of **b** onto **a**

Figure 6.11: The projection of vector **a** onto vector **b** and vice versa.

b is written:

$$\mathbf{a} \cdot \mathbf{b} = ab \cos\theta \tag{6.5}$$

Equation (6.5) demonstrates that the scalar product, or dot product (since a dot is used to indicate the multiplication operation), of vector **a** and vector **b** is equal to the magnitude of one of the vectors multiplied by the component of the other vector in the direction of the first. Refer to Figure 6.11a. If **b** is the direction we are going, then the component of vector **a** in the direction of **b** is simply the trigonometric function $a\cos\theta$. So the dot product is $ba\cos\theta$, or rearranged, $ab\cos\theta$. Refer now to Figure 6.11b: If **a** is the direction we are going, then the component of vector **b** in the direction of **a** is simply the trigonometric function $b\cos\theta$. So regardless which vector is chosen as the base direction, the dot product is $ab\cos\theta$.

In terms of components, the dot product is determined as:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Example 6.1. Consider the diagram of Figure 6.12, in which a force vector **F** is applying a pull of 20 pounds (88.8 newtons) on a box at an angle of 30 degrees from the horizontal. If the box moves at a constant velocity and undergoes a displacement **d** of 8 feet (2.43 meters) to the right, how much work is done?

From physics we have the formula for work:

$$work = W = \mathbf{F} \cdot \mathbf{d}$$

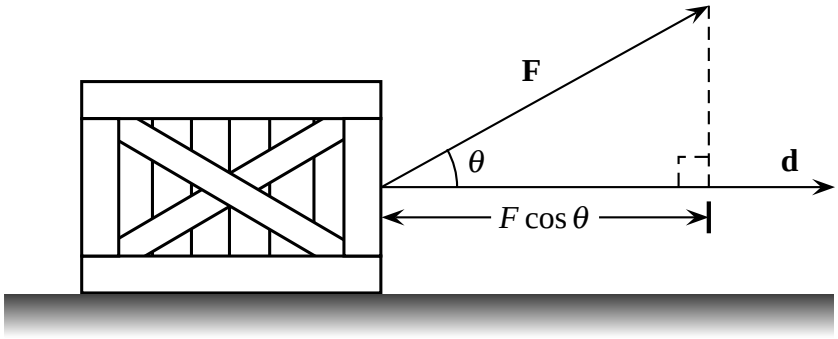


Figure 6.12: An example of using the dot product to determine the work done after moving a box.

which is the dot (scalar) product of **F**, the force vector being applied, and **d**, the displacement vector of the object. This equation is used when the force and the displacement are in the same direction. In this example, however, the force is at a 30 degree angle to the displacement. Thus, the component of the force in the direction of the displacement must be used. This component is defined by the trigonometric function $F\cos\theta$. Substituting this function into the work equation and then substituting in the values yields:

US	SI
$W = d(F\cos\theta)$	$W = d(F\cos\theta)$
$= 8(20\cos 30^\circ)$	$= 2.43(88.8\cos 30^\circ)$
$= 8(20)(0.866)$	$= 2.43(88.8)(0.866)$
$= 138.56$ ft-lbs	$= 186.86$ N-m or J

6.6.3 Cross Product

Two vectors can be multiplied together with a vector product as a result. Consider two vectors **a** and **b** and the angle θ between them in Figure 6.13. The vector product of **a** and **b** is written $\mathbf{a}\times\mathbf{b}$. Thus:

$$\mathbf{c} = \mathbf{a}\times\mathbf{b}$$

This equation demonstrates that the vector product, or cross product (since a cross is used to indicate the multiplication operation), of vector **a** and vector **b** is another vector equal to $\mathbf{a} \times \mathbf{b}$.

In terms of components:

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

Now since **c** is a vector, it has magnitude and direction. Thus the magnitude of **c** is defined by $c = ab\sin\theta$, and the direction of **c** is defined to be perpendicular to the plane formed by vectors **a** and **b**. The sense of vector **c**, however, is found by the right-hand rule.

Consider Figure 6.13: To determine the sense of vector **c**, imagine vector **a** rotating toward vector **b** through the smaller angle θ between them (the smaller of the two angles is always chosen in vector multiplication). Now take your right hand and curl your fingers in the direction of the rotation of **a** toward **b**. Extend your thumb and it will point in the direction (sense) of vector **c**.

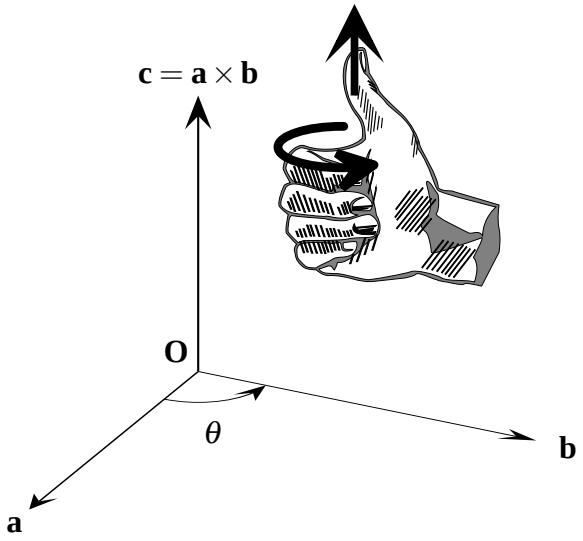


Figure 6.13: Pictorial representation of a cross product for which we can using the right-hand rule to determine the direction. (View in three dimensions.)

This right-hand rule can also be seen in action when a screw is turned into a block of wood: If the screw is turned in the direction of the curled fingers of the right hand, the screw will advance in the direction of the extended thumb.

When exploring dot products, we saw that the order in which the vectors were multiplied did not matter. However, this is not true for cross products: $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$. For $\mathbf{b} \times \mathbf{a}$, the magnitude will be the same as for $\mathbf{a} \times \mathbf{b}$, but the direction will be opposite, as shown in Figure 6.14.

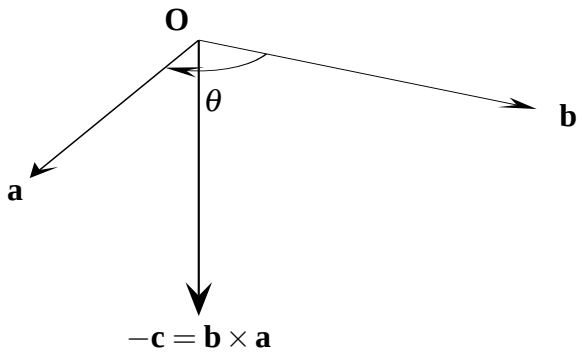


Figure 6.14: Cross products do not commute because $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$.

An application in which a cross product is used was briefly introduced in Section 6.2. That example involved removing a nut with the aid of a wrench, as in Figure 6.3 on page 6-4.

In Figure 6.3a, a force is being applied to the end of a wrench in an effort to turn the nut counter-clockwise to remove it. This force on the wrench produces a quantity called *torque*, which is the measure of the tendency of a body to rotate under the action of forces.

This action is represented pictorially in Figure 6.3b. Figure 6.3b is an intuitive representation of what is occurring: The force vector **F** is being applied to the end of the position vector **L**, that is, to the end of the wrench, to get the nut to turn. The force vector **F** is perpendicular to **L**. However, the analytical representation of this action for computational purposes is seen in Figure 6.3c, in which the position vector and force vector, in that order, are shown at point **O**, on which they both operate.

Continuing the analysis, consider Figure 6.15 on page 6-16, which expands Figure 6.3c. If vector **L** rotates toward vector **F**, the cross product $\mathbf{L} \times \mathbf{F}$ is being performed on the two vectors. As previously seen, the result of a cross product of two vectors is also a vector and is perpendicular to the plane formed by the original vectors. Thus the vector resulting from the cross product $\mathbf{L} \times \mathbf{F}$ is perpendicular to the plane formed by **L** and **F** and is called the moment of the force **F** with respect to point **O**. This moment is written $\mathbf{M_O}(\mathbf{F})$. The direction of the moment is defined by the right-hand rule and in this example is in the direction necessary to remove the nut (remember the old adage: “left to loosen, right to tighten”).

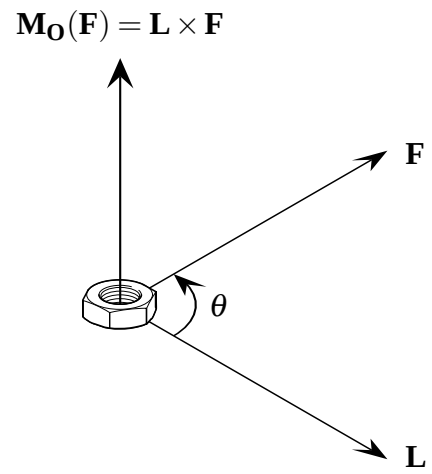


Figure 6.15: The torque (moment) vector acting on a nut.

6.7 Converting Polar Coordinates to Rectangular Coordinates

Polar coordinates can be converted to rectangular coordinates by using trigonometric functions. With reference to Figure 6.16, consider the point located by polar coordinates $\{r, \theta\}$. In rectangular coordinates, the point would be located some distance in the x-direction and some distance in the y-direction and would have the coordinates (x, y). If the x and y distances are shown on the diagram along with distance r, a right triangle is formed. Thus, the values for x and y can be determined through the use of trigonometric functions.

Since θ is the angle of interest, the following trigonometric relationships can be utilized:

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

Substitute x, y, and r for adjacent, opposite, and hypotenuse, respectively:

$$\cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r}$$

Solving each equation for x and y yields the following equations:

$$x = r\cos\theta \qquad y = r\sin\theta$$

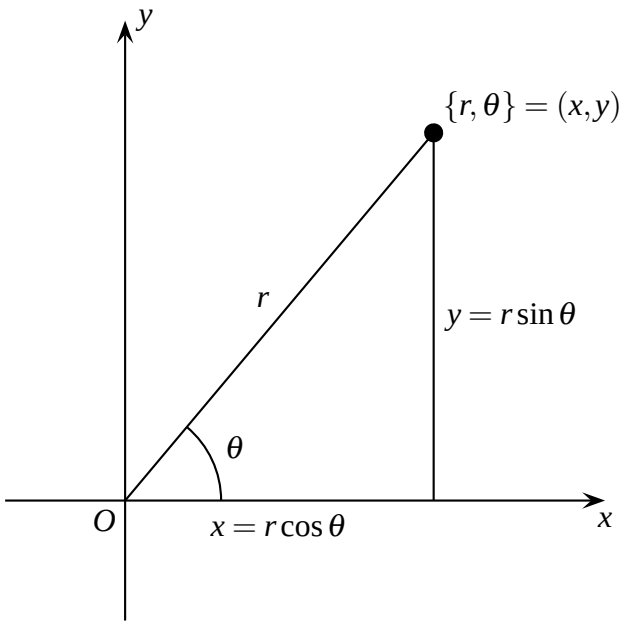


Figure 6.16: An illustration to show conversion of coordinates.

which convert polar coordinates to rectangular coordinates.

6.8 Resolving a Vector into Components

Resolving vectors into components uses the same technique of converting polar coordinates to rectangular coordinates.Rectangular coordinates can be converted to polar coordinates by using the Pythagorean Theorem. Reconsider Figure 6.16; look the point located by rectangular coordinates (x,y). In polar coordinates, the point would be located some distance r from the origin in the direction defined by angle θ and would have the coordinates $\{r,\theta\}$. If the x and y distances are shown on the diagram, they become the sides of a right triangle with distance r as the hypotenuse.

Thus, the value of r can be determined from the Pythagorean Theorem:

$$r = \sqrt{x^2 + y^2}$$

and once r has been determined, the angle θ can be found by first using either of the following

To convert polar coordinates $\{r,\theta\}$ to rectangular coordinates (x,y) use:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

To convert rectangular coordinates (x,y) to polar coordinates $\{r,\theta\}$ use:

$$r = \sqrt{x^2+y^2}$$
$$\theta = \cos^{-1}\frac{x}{r} \quad \text{or} \quad \theta = \sin^{-1}\frac{y}{r}$$

Table 6.2: Converting between Polar and Rectangular Coordinates

trigonometric functions:

$$\cos\theta = \frac{x}{r} \quad \text{or} \quad \sin\theta = \frac{y}{r}$$

and then solving for θ with either of these functions:

$$\theta = \cos^{-1}\frac{x}{r} \quad \text{or} \quad \theta = \sin^{-1}\frac{y}{r}$$

Using the trigonometric function $\tan\theta = \frac{y}{x}$ to determine the angle θ would not be sufficient since it wouldn't distinguish the quadrant of the angle. The tangent values are the same in quadrants I and III, as well as in II and IV.

In Chapter 12 we will be drawing a vector diagram to analyze a crash. This is an example of the use of polar coordinates. In plotting the departure vectors of Unit #1 and Unit #2, the magnitudes v_3 and v_4 are the r values for the units and the departure angles are the θ values for the units. A similar correspondence exists with respect to the approach vectors of the units.

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6.8.1 Example

Determine the components of a vector that has a magnitude of 62 pointing NNE. Consider East to be the x-axis.

The vector is pointing half way between N and NE. This corresponds to an angle of 67.5° from

East (the x-axis). The x component is found by using the cosine function:

$$\begin{aligned} x &= r\cos\theta \\ &= 62\cos 67.5 \\ &= 23.72 \end{aligned}$$

Similarly, the y component is determined using the sine function:

$$\begin{aligned} y &= r\sin\theta \\ &= 62\sin 67.5 \\ &= 57.28 \end{aligned}$$

Example 6.2. Determine the x and y components of a vector that has a magnitude of 30 pointing in a direction that is 200° from the x-axis.

$$\begin{aligned} x &= r\cos\theta \\ &= 30\cos 200 \\ &= -28.19 \end{aligned}$$

$$\begin{aligned} y &= r\sin\theta \\ &= 62\sin 200 \\ &= -10.26 \end{aligned}$$

Notice both components are negative. This means that if the tail of the vector was placed at the origin, then the tip of the vector would be in the third quadrant.

Exercises

1. Vector **A** has a magnitude of 10 units and a direction of 0° with respect to the *x*-axis of a coordinate axis system. Vector **B** has a magnitude of 15 units and a direction of 45° with respect to the same axis.

   10 units — a) What is the magnitude of the *x* component of vector **A**?

   10.6 units — b) What is the magnitude of the *x* component of vector **B**?

   0 units — c) What is the magnitude of the *y* component of vector **A**?

   10.6 units — d) What is the magnitude of the *y* component of vector **B**?

   23.16 units — e) What is the magnitude of the sum of vectors **A** and **B**?

   27.22° — f) What is the angle of the sum of vectors **A** and **B**?

2. A vector has coordinates $(x, y, z) = (4, 5, 10)$.

   11.87 — a) What is its length?

   57.37° — b) What angle does it make with the *xy* plane?

   32.63° — c) What angle does it make with the *z*-axis?

   10 — d) What is its component in the *z*-direction?

   6.4 — e) What is the length of its component in the *xy* plane?

   51.34° — f) What is the angle from the *x*-axis of the vector component in the *xy* plane?

3. A vector, **A**, has a length of 1500 units and an angle of 35° with respect to the *x*-axis. What is the result of subtracting a vector, **B**, of length 300 units and direction of 195° with respect to the *x*-axis from vector **A**?

   1784 units
   31.7°

4. If we pull a concrete filled tire on a level surface with a force of 100 lb and an angle of 25°,

   90.63 lb — a) What is the force parallel to the ground (dragging the tire)?

   42.26 lb — b) What is the force perpendicular to the ground (lifting the tire)?

5. The torque used to turn a bicycle sprocket is generated by the weight of the rider acting through the lever arm of the crank. If a bicycle crank is 25 cm from shaft to pedal and the rider pushes with 400 N, then determine the torque when the pedal is at 45°. Let the *x*-axis be forward and the *y*-axis be straight up. *(Hint: Take the cross product of the force vector and the radius vector.)*

N-cm

a) What is the magnitude of the torque?

b) What is the direction of the torque?

To the rider's lef

7 Motion In One Dimension

This chapter is devoted to understanding the concepts of motion in a straight line. This is a specific case of real-world motion, but it arises frequently enough in traffic crash reconstruction to warrant its own chapter. Also, a good grasp of the concepts being presented in this chapter will enable the reader to understand motion in two or three dimensions as described in Chapter 8.

7.1 The Study of Motion: Kinematics

Kinematics is the study of motion without regard to what causes the motion. A kinematic analysis merely looks at the motion of one object with respect to another. A kinematic observation is that a free falling body accelerates at a constant rate, regardless of its size or weight. This observation will enable us to calculate other information like speed and distance, but does not explain why the acceleration is constant. We define some concepts of kinematics below; the definition of the units for these quantities are defined in Chapter 5.

Distance is a measure of how far an object has traveled. The measure of distance traveled always considers where an object began. In other words, it is necessary to be able to retrace the path of an object in order to compute distance. Distance is a scalar quantity and is the magnitude of displacement.

Displacement is a measure of the straight line distance from one point to another. Displacement is a vector quantity because the direction of the measured distance is important. The magnitude of displacement is distance.

Speed is a measure of how fast an object is going in a straight line. The speed of an object is a scalar quantity and it does not matter which direction the object is traveling. The average speed of an object is the distance traveled divided by the time it took to travel that distance.

Velocity is a vector representing both the speed and direction of an object. The average velocity of an object is the displacement traveled divided by the time it took to travel that displacement.

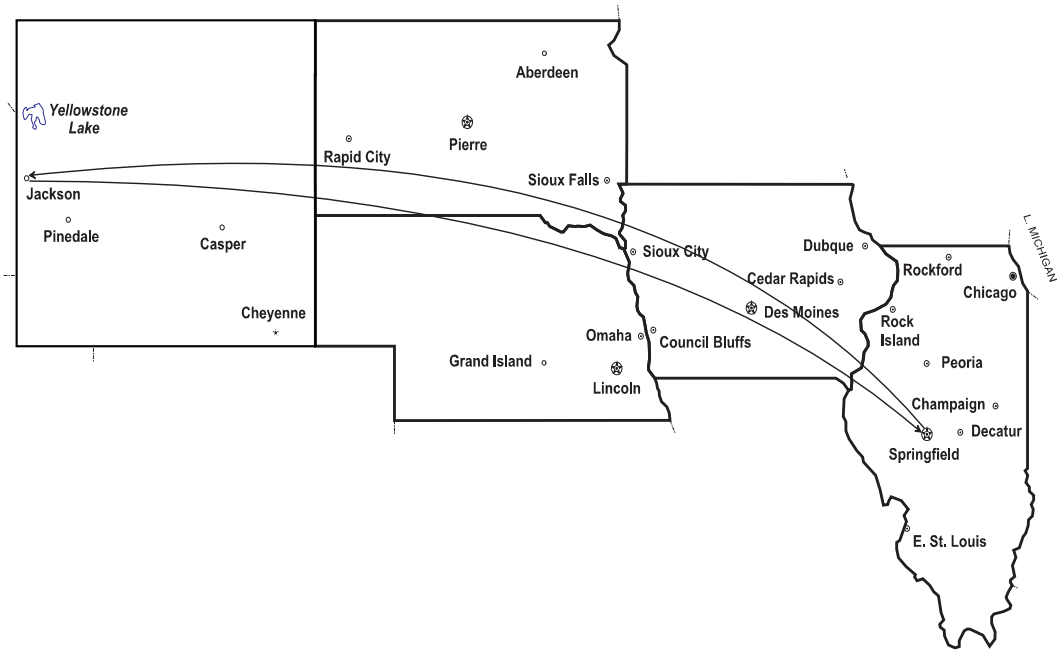


Figure 7.1: The traveler who begins and ends in Springfield, IL undergoes no displacement, but travels a distance of over 2000 miles.

Since displacement is a vector and time is a scalar, velocity must also be a vector. Velocity is the first derivative of displacement with respect to time.

Despite what many people think, there is a difference between velocity and speed. Velocity is a vector quantity, whereas speed is a scalar quantity. What does this mean? A vector quantity is any quantity that must be described by both a magnitude and a direction, whereas a scalar quantity may be described simply by a magnitude. In the case of a vehicle, we might say that its speed is 55 mph with respect to the road surface. However, to describe the velocity of the vehicle, we would also have to specify the vehicle's direction. We would have to say, for example, that the vehicle is traveling 55 mph due north. To deal effectively with some traffic crash problems, we must know the velocity of the vehicle rather than just its speed.

Acceleration is the rate of change of velocity with respect to time. Since velocity has both magnitude and direction, acceleration occurs when there is a change in either the magnitude or direction of velocity. Since velocity is a vector, so is acceleration. Therefore, acceleration always has a direction associated with it. Acceleration is the first derivative of velocity with respect to time and the second derivative of displacement with respect to time.

Deceleration is a common term used that is not definable in a general sense. The term deceleration is taken to mean a loss of speed. In a mathematical sense, deceleration is actually a

negative acceleration. If motion is restricted to a straight line (one dimension) then deceleration is any acceleration that is opposite the velocity vector.

Jerk is the time rate of change of acceleration. Jerk is the first derivative of acceleration with respect to time and the third derivative of displacement with respect to time. The reason we define it here is because air bag modules look for jerk when making decisions about deploying an air bag.

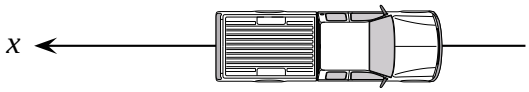
7.2 Rectilinear Motion

When we discuss the movement of vehicles in the following sections, it will be assumed that they are on the ground. Further, we will limit our discussion to the straight-line motion of these vehicles. This eliminates the need to consider two of the three axes and focuses our study on the movement of vehicles in one dimension. This one-dimensional motion is called *rectilinear motion* and is generally chosen to be along the x-axis.

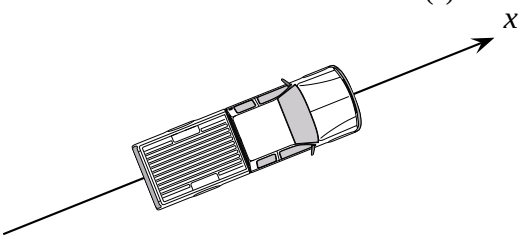
We know that velocity and acceleration are vector quantities, which possess both magnitude and direction. Since our discussion is limited to one dimension (the x-axis), the direction component of all vector quantities (displacement, velocity, and acceleration) is determined by their sign. So the vector quantities can now be represented as one value with the negative value being the opposite direction. Thus, in the ensuing discussion, the symbol for velocity does not have to be shown in the format of a vector, $\mathbf{v}$ or $\bar{v}$, but can be shown in the format of a variable, v_x, or just v. Similarly, the symbol for acceleration does not have to be shown in the format of a vector, $\mathbf{a}$ or $\bar{a}$, but can be shown in the format of a variable, a_x, or just a.

Displacement is also a vector. However, as with velocity and acceleration, its direction is restricted to the x-axis so the vector quantity is represented as its magnitude (distance) with a sign. Negative values of displacement are opposite in direction from positive displacements. Only in the specific case of one-dimensional motion can the terms displacement and distance be interchanged (since the direction is fixed). In traditional physics texts and literature, the variable x or s is used to represent the amount of change in the position of an object. However, in this chapter we will use the variable d instead, as it is more familiar to crash investigators.

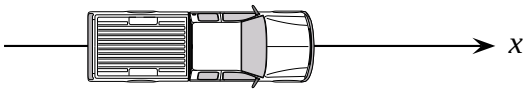
The first step in solving problems in kinematics is to define a coordinate system. In the case of rectilinear motion, one positive direction needs to be defined. Figure 7.2 shows different rectilinear coordinate systems. Once a system is defined, the motion described is in relation to that coordinate system.



(a) Positive motion is from right to left



(b) Positive motion is diagonal up and right



(c) Positive motion is from left to right

Figure 7.2: Different rectilinear coordinate systems for the straight line motion of a pickup truck. Notice that if the pickup in subfloat (a) were to travel in a positive direction, it would have to be going backwards.

7.2.1 Constant Velocity

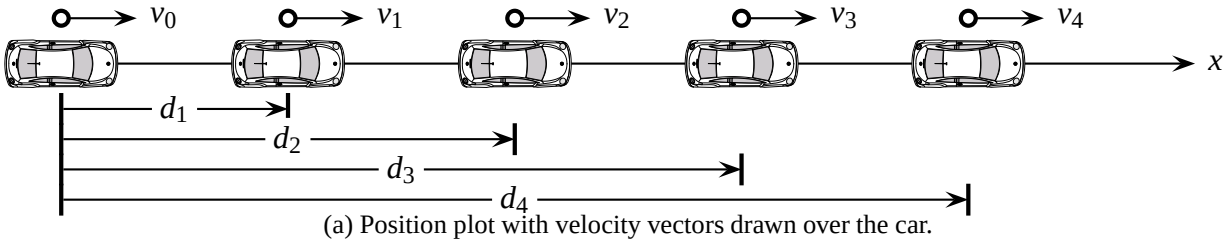
If the average velocity of an object does not change between two points along a path, then its velocity is considered constant. This also means the velocity does not change over a given time period. The result of this is that the path traveled will be a straight line and the motion will take place at a uniform rate (constant speed).

Figure 7.3 on page 7-5 shows pictorially what it means for an object to have constant velocity. The position plot in Figure 7.3a illustrates how the motion of the vehicle is in a straight line. Each rendering of a vehicle represents a snapshot in time. Notice how each snapshot is equally spaced and has the same instantaneous velocity vector. The points on the time-history graphs are individual snap shots in time and are denoted with a subscripted number. For example, the vehicle motion at time t_2 is highlighted on the graph to help us understand how to read the graph.

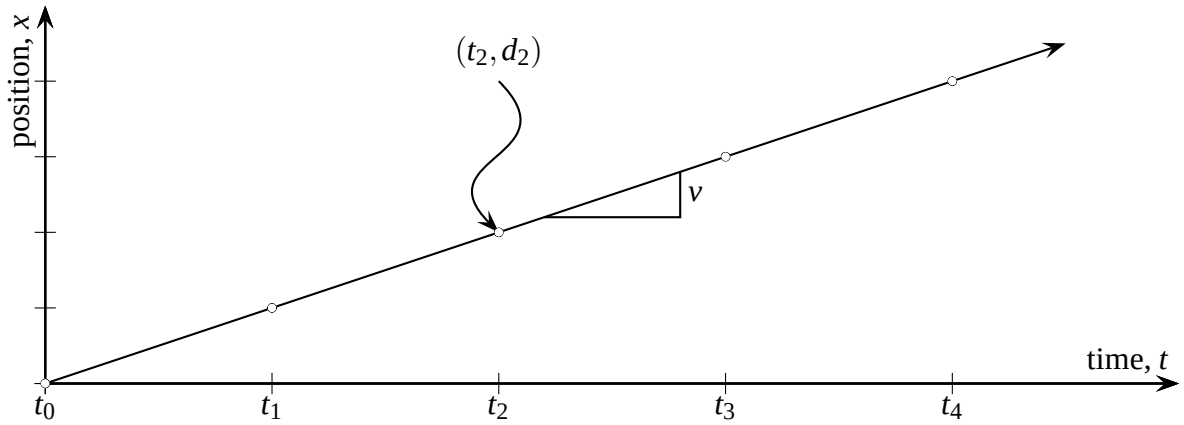
An *instantaneous* velocity is the exact velocity at a particular moment of time. There is no averaging used to obtain the instantaneous velocity, however, in the case of constant velocity, the instantaneous velocity is equal to the average velocity. Referring again to Figure 7.3, we can see that

$$v = v_0 = v_1 = v_2 = v_3 = v_4$$

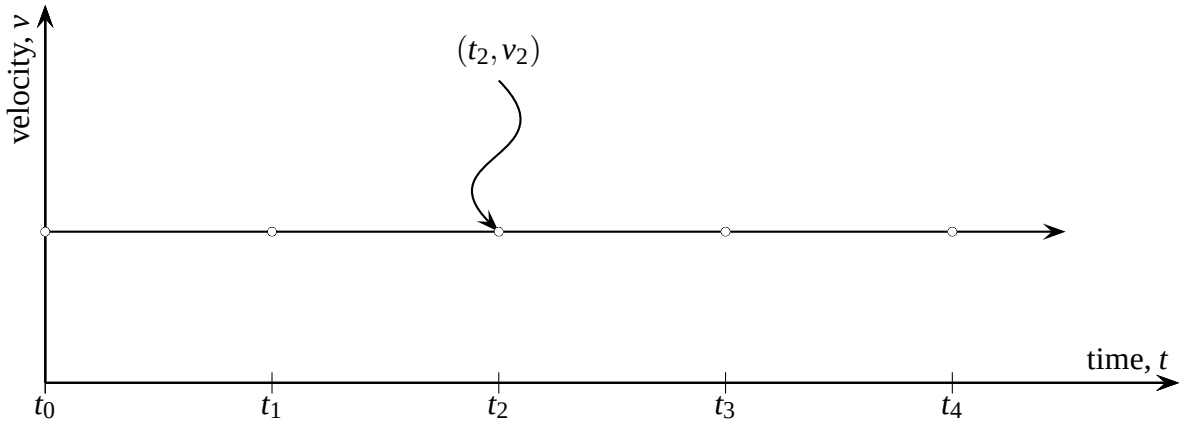
Figure 7.3b shows the plot of the position against time for constant velocity. This graph is a line



(a) Position plot with velocity vectors drawn over the car.



(b) Time history of position with respect to time.



(c) Time history of velocity with respect to time.

Figure 7.3: The position plot and time histories of a vehicle moving at constant velocity. The points in the position plot (a) are equally spaced in time.

with a constant slope and the slope is equal to the velocity. From the definition of slope we know that the velocity must be the ratio of the change of position to the change in time:

$$v = \frac{\Delta d}{\Delta t} \tag{7.1}$$

Since we are dealing with constant velocity, Eq. (7.1) is used to determine the velocity given a change in time and a change in distance. So the question remains, how do we determine the change in time and distance? The answer: We must have two points where we know both the time and distance, then we use subtraction to figure out the change.

$$v = \frac{\Delta d}{\Delta t} = \frac{d_2 - d_1}{t_2 - t_1} \tag{7.2}$$

Point 2 is generally taken to be the point with the greater time and point 1 can sometimes be zero. This is important because both velocity and displacement are vectors and the sign indicates the direction. Since Δt is a scalar, having a negative time does not make as much sense as having a vector quantity point in the opposite direction. Obviously, the two times in Eq. (7.2) cannot be equal.

To determine a change in displacement* over a specified time while experiencing constant velocity, rearrange Eq. (7.1):

$$\Delta d = v\Delta t \tag{7.3}$$

which simply states that an object moving with a constant velocity v for a period of time Δt will cover a distance of Δd.

Similarly, determining the time to cover a specific distance is done by rearranging Eq. (7.1):

$$\Delta t = \frac{\Delta d}{v} \tag{7.4}$$

which says that if a certain distance d is covered while an object is moving or traveling at a constant velocity v, it will take time t.

*Since we are dealing with motion in one dimension, the change in displacement is equal to the distance traveled.

7.2.2 Constant Acceleration

When observing the movement of vehicles, it becomes readily apparent that they do not travel at constant velocity very long. Vehicles will speed up, slow down, or change direction. When this occurs, the vehicle has been *accelerated*. However, since we are limiting our discussion to one dimension, we will be concerned with the speeding up and slowing down action only. Acceleration due to a change in direction will be discussed in Section 14.1. We will further limit our discussion to a constant acceleration, in which the average acceleration for any time interval is equal to the instantaneous acceleration. Constant acceleration is commonly assumed for vehicle dynamics and, for our purposes, the acceleration due to gravity is always constant.

Constant acceleration of an object (also known as uniformly accelerated motion) produces a uniform (linear) change in velocity, as illustrated in Figure 7.4 on page 7-8.

The illustrations in Figure 7.4 show us how the velocity is now changing. This change is at a constant rate and produces a change in displacement that follows a parabola. A constant acceleration leads to a velocity that changes in a linear fashion with a constant slope.

Acceleration is defined as a change in velocity with respect to time, as demonstrated by the slope of the line in Figure 7.4c. Again, from the definition of slope:

$$a = \frac{\Delta v}{\Delta t} = \frac{v_f - v_o}{t_f - t_o} \tag{7.5}$$

where the subscript $_f$ denotes the final point and the subscript $_o$ denotes the initial point. Usually, when a time interval is being measured, the timing starts at 0 seconds: Thus, $t_o = 0$ and Eq. (7.5) can be simplified to:

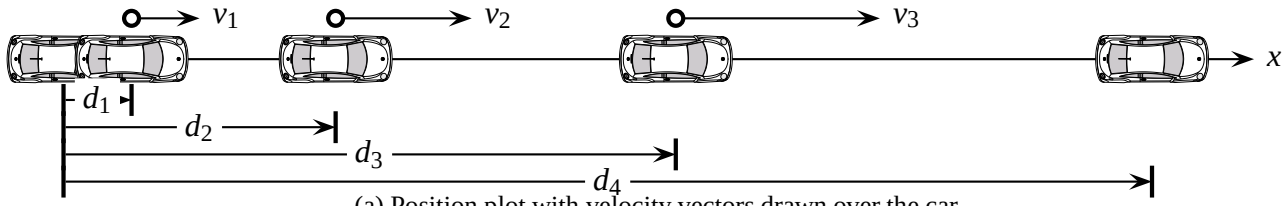
$$a = \frac{v_f - v_o}{t} \tag{7.6}$$

where $t = t_f$. Further, if the vehicle was originally stopped, i.e. $v_o = 0$, then Eq. (7.6) becomes

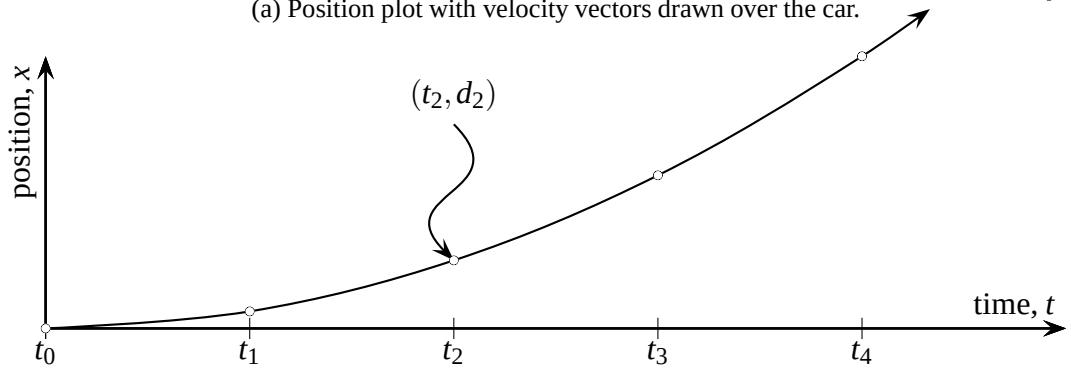
$$a = \frac{v}{t} \tag{7.7}$$

where $v = v_f$.

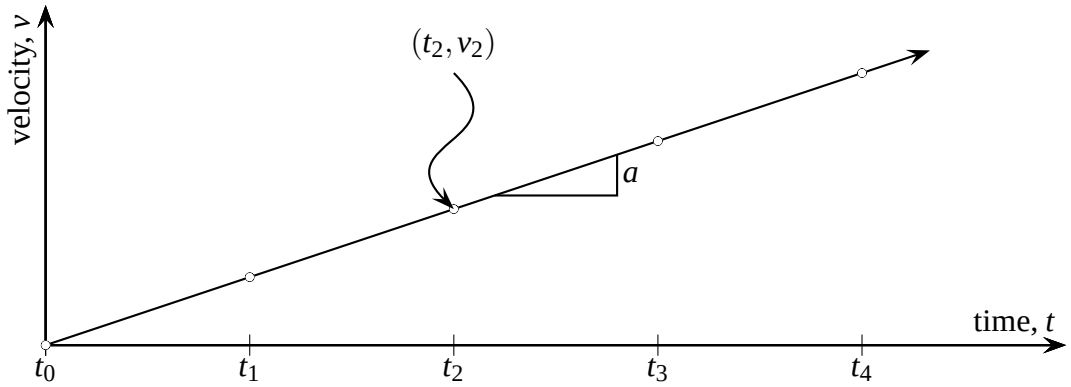
As we did with velocity, we can manipulate Eq. (7.5) to solve for the other variables. Multiplying



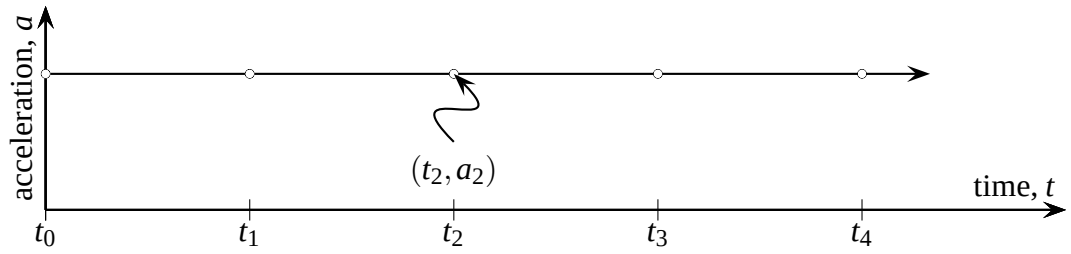
(a) Position plot with velocity vectors drawn over the car.



(b) Time history of position with respect to time.



(c) Time history of velocity with respect to time.



(d) Time history of acceleration with respect to time.

Figure 7.4: The position plot and time histories of a vehicle accelerating at a constant rate. The points in the position plot are equally spaced in time, whereas the spacing of the positions of the vehicles are no longer equal.

both sides of the equation by t and rearranging yields the following equation for velocity:

$$\Delta v = at \tag{7.8}$$

which simply demonstrates that if an object, or vehicle, accelerates with constant acceleration for a period of time t, it will experience a change in velocity by the amount Δv. Note that this change in velocity Δv does not consider original velocity or final velocity.

If Eq. (7.8) is divided by the acceleration a and rearranged, the following equation for time t is obtained:

$$t = \frac{\Delta v}{a}$$

which demonstrates that if a certain velocity change Δv is divided by the rate a, then it will take time t for this change to take place.

7.3 Uniformly Accelerated Motion

Having defined expressions for velocity and constant acceleration, expressions can now be developed that will define the relationships between acceleration, velocity, time, and displacement. These are known as kinematic relationships. These expressions can then be used to solve problems involving uniformly accelerated motion. All expressions derived for this section come from two basic definitions defined in Eqs. (7.1) and (7.5). From here on, we will assume that $t_0 = 0$ so $\Delta t = t_f$ will be shortened to simply t to denote the time interval.

7.3.1 Relationship Between Acceleration and Velocity

Recall Eq. (7.5):

$$a = \frac{\Delta v}{\Delta t} = \frac{v_f - v_o}{t_f - t_o} = \frac{v_f - v_o}{t}$$

which defines acceleration. It can be solved to find v_f, the final velocity of a vehicle after accelerating for time t. Start by multiplying both sides by t:

$$at = \frac{v_f - v_o}{\not{t}} \not{t}$$
$$at = v_f - v_o$$

Add v_o to both sides:

$$at + v_0 = v_f - v_o + v_o$$
$$at + v_o = v_f$$

Rearrange:

$$v_f = v_o + at \tag{7.9}$$

which says:

$$\text{Final velocity} = \text{Initial velocity} + \text{Velocity change}$$

Thus, if a vehicle is traveling with an initial velocity, v_o, and begins to accelerate, the velocity of the vehicle will change by the amount $a\Delta t$ and the vehicle will be traveling at a new velocity, v_f, at the end of the time interval Δt.

The acceleration rate a can be either positive $(+)$ or negative $(-)$ depending on whether the vehicle is speeding up or slowing down. Often, the time interval Δt is defined to start from zero ($t_o = 0$), in which case $\Delta t = t_f$.

7.3.2 Relationship between Acceleration, Time, and Displacement

When a vehicle moves, it obviously covers some distance d. If we look at a finite time interval for this motion, say from some initial time t_o to a final time t_f, we can see that the vehicle will be traveling at some initial velocity v_o at time t_o and at some final velocity v_f at time t_f.

If the vehicle is traveling at a constant velocity, its time-history graph looks like this:

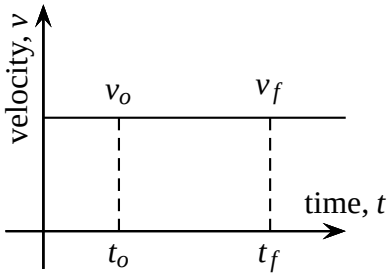


Figure 7.5: Constant velocity.

And if it is undergoing constant acceleration, its time-history graph looks like this:

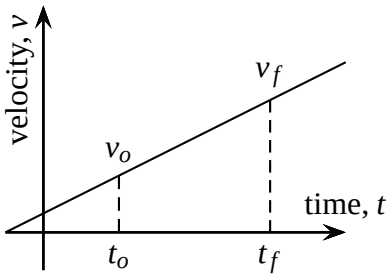


Figure 7.6: Constant acceleration.

In either case, the graph is linear and the average velocity during the time interval can be found the same way as the average of any two numbers is found. Add them and divide by 2:

$$v_{avg} = \frac{v_o + v_f}{2} \tag{7.10}$$

The distance traveled during this time interval can be described by the equation

$$d = v_{avg}\Delta t \tag{7.11}$$

For the constant velocity situation, v_{avg} is simply the value of the constant velocity for any interval in time. However, for the constant acceleration situation, v_{avg} changes with each time interval and a new expression for the distance covered during each new time interval must be developed. This new expression must take into account the distance covered during acceleration.

We can substitute Eq. (7.9) on page 7-10 into Eq. (7.10):

$$\begin{aligned} v_{avg} &= \frac{v_o + v_f}{2} \\ &= \frac{v_o + v_o + at}{2} \\ &= \frac{2v_0 + at}{2} \\ v_{avg} &= v_o + \frac{1}{2}at \end{aligned}$$

Now substitute this expression for v_{avg} into Eq. (7.11):

$$\begin{aligned} d &= v_{avg}t \\ &= \left(v_o + \frac{1}{2}at\right)t \\ d &= v_o t + \frac{1}{2}at^2 \end{aligned} \tag{7.12}$$

This is the expression we are looking for. It states that:

$$\text{distance} = \text{distance traveled at constant velocity} + \text{additional distance traveled during acceleration.}$$

7.3.3 Relationship between Acceleration, Velocity, and Displacement

Equation (7.5) is an expression for velocity that involves acceleration and time. We can also find an expression for velocity that involves acceleration and displacement instead of time.

Start with the basic formula for acceleration:

$$a = \frac{v_f - v_o}{t}$$

Solve it for time. First multiply both sides by time, then divide both sides by a:

$$\begin{aligned} at &= v_f - v_o \\ \frac{\cancel{a}t}{\cancel{a}} &= \frac{v_f - v_o}{a} \\ t &= \frac{v_f - v_o}{a} \end{aligned}$$

Now substitute this expression for time into the displacement/time formula of Eq. (7.12):

$$\begin{aligned} d &= v_o t + \frac{1}{2}at^2 \\ d &= v_o\left(\frac{v_f - v_o}{a}\right) + \frac{1}{2}a\left(\frac{v_f - v_o}{a}\right)^2 \end{aligned}$$

Multiply through the first term by v_o and begin expanding the second term:

$$d = \frac{v_o v_f - v_o^2}{a} + \frac{1}{2}a\left(\frac{v_f - v_o}{a}\right)\left(\frac{v_f - v_o}{a}\right)$$

Expand the second term further by applying the FOIL method from Section **??**:

$$d = \frac{v_o v_f - v_o^2}{a} + \frac{1}{2}a\left(\frac{v_f^2 - 2v_o v_f + v_0^2}{a^2}\right)$$

Cancel the a in the numerator and one a in the denominator of the second term and multiply through by the 2 in the denominator:

$$d = \frac{v_o v_f - v_o^2}{a} + \left(\frac{v_f^2 - 2v_o v_f + v_0^2}{2a}\right)$$

Find a common denominator and group terms:

$$
\begin{aligned}
d &= \frac{v_o v_f}{a} - \frac{v_o^2}{a} + \frac{v_f^2}{2a} - \frac{2v_o v_f}{2a} + \frac{v_0^2}{2a} \\
&= \underbrace{\frac{2v_o v_f}{2a}}_{cancel} - \frac{2v_o^2}{2a} + \frac{v_f^2}{2a} - \underbrace{\frac{2v_o v_f}{2a}}_{cancel} + \frac{v_0^2}{2a} \\
&= \frac{v_f^2}{2a} + \frac{v_0^2}{2a} - \frac{2v_o^2}{2a} \\
d &= \frac{v_f^2 - v_o^2}{2a}
\end{aligned}
$$

Now solve for v_f by multiplying both sides by $2a$:

$$2ad = v_f^2 - v_o^2$$

Add v_o^2 to both sides and rearrange:

$$v_f^2 = v_o^2 + 2ad \tag{7.13}$$

Take the square root of both sides to obtain the desired expression:

$$v_f = \sqrt{v_o^2 + 2ad} \tag{7.14}$$

Constant Velocity

Velocity	$v = \dfrac{d}{t}$
Displacement	$d = vt$
Time	$t = \dfrac{d}{v}$

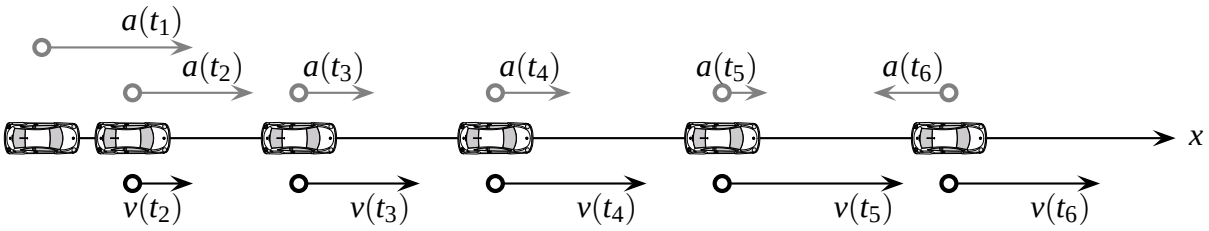
Constant Acceleration (Uniformly Accelerated Motion)

Acceleration	$a = \dfrac{v_f - v_o}{t}$
Velocity	$v_f = v_o + at$ $v_f = \pm\sqrt{v_o^2 + 2ad}$
Displacement	$d = \left(\dfrac{v_o + v_f}{2}\right) t$ $d = v_o t + \dfrac{1}{2}at^2$

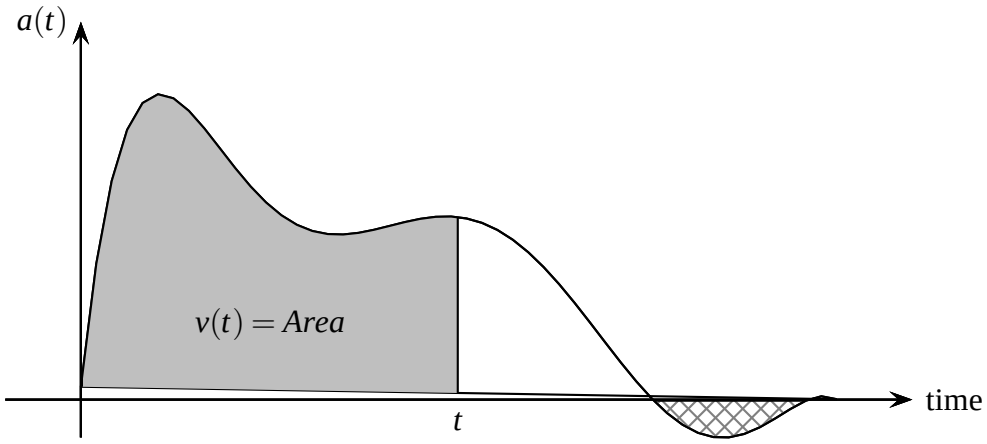
Table 7.1: Summary of Equations for Rectilinear Motion

7.4 Variable Acceleration

In the previous section we discussed the motion of a particle undergoing constant acceleration. Often times, however, an object will experience a variable acceleration, which means that the rate of acceleration varies with respect to time. The graph of the varying acceleration will be nonlinear.



(a) Position of a vehicle at specific times. Acceleration vectors are on top in gray and velocity vectors are on the bottom in black.



(b) Time history of the acceleration of the vehicle during its travel. The area under the acceleration curve is the velocity.

Figure 7.7: A vehicle accelerates from a stop and reaches a cruising speed. Let the starting point of the car be the origin and the direction of travel is the positive x axis in subfloat (a).

We represent a time-varying acceleration as a function of time. It is written in function notation, $a(t)$, which is different from multiplying acceleration by time, at.

7.4.1 Kinematic Relations Using Calculus Concepts

To illustrate the concepts of variable acceleration, consider an example of a car accelerating from a stop and reaching a cruising speed. The motion diagram of Figure 7.7a shows the positions of the vehicle at different points in time. A graph of the acceleration time history is shown in Figure 7.7b. Notice that there are both positive and negative accelerations. Also, when the car is cruising at a constant velocity, the acceleration is zero.

Now the question remains; how far did the vehicle go and how fast was it going at any particular time? To answer this, we will use a property of velocity that comes from calculus that says the area under the acceleration curve plus the initial velocity is the final velocity at that instant. If the initial velocity is zero, then the final velocity is the area under the curve. This is the process of integration

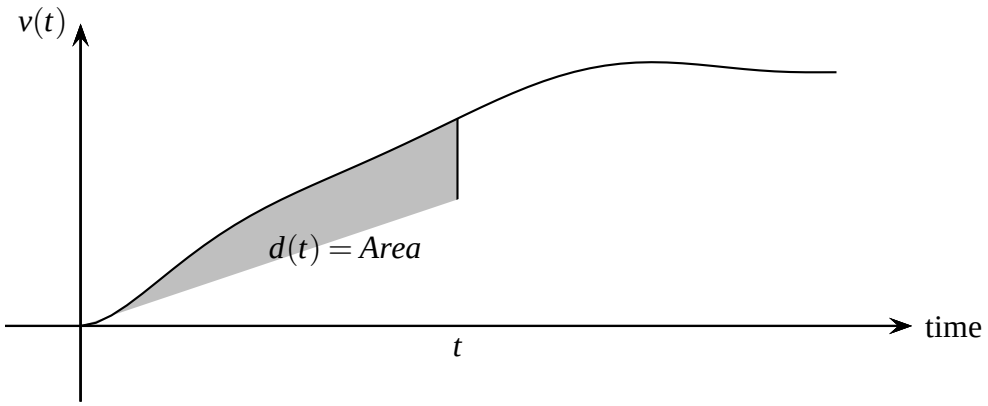


Figure 7.8: The time history of the displacement is obtained by determining the area under the velocity curve.

and is written as:

$$v(t) = v_o + \int_{t_o}^{t} a(s)\,ds \tag{7.15}$$

What this means is that the velocity at time t, written as $v(t)$, is the initial velocity plus the area under the acceleration curve from time t_o to time t, which is the shaded area shown in Figure 7.7(b). You may be wondering what the variable s is in Eq. (7.15). The variable s is a place holding variable because we have already used t in the limits of integration. It is called a *dummy variable* and is necessary to make the expression mathematically correct. For every value of t, the process must be repeated to find the velocity for each time t. This is not a hard process when the acceleration is constant because we can use formulas for areas of triangles and rectangles to find the area under the curves (lines). This is a numerical approximation of the area under the curve.

When the acceleration is negative, as shown in the cross-hatched region of Figure 7.7(b), the vehicle is slowing down. When the acceleration is negative, then the area is considered to be negative. It does not make physical sense to have a negative area, but mathematically it is necessary to let the region below the t axis have a negative area. The velocity of an object will increase as long as the acceleration continues to be positive.

The next step is to determine the displacement of the vehicle. Displacement at time t is written as $d(t)$ and is equal to the initial displacement d_o plus the area under the velocity curve. Again, this is written as an integral:

$$d(t) = d_o + \int_{t_o}^{t} v(s)\,ds \tag{7.16}$$

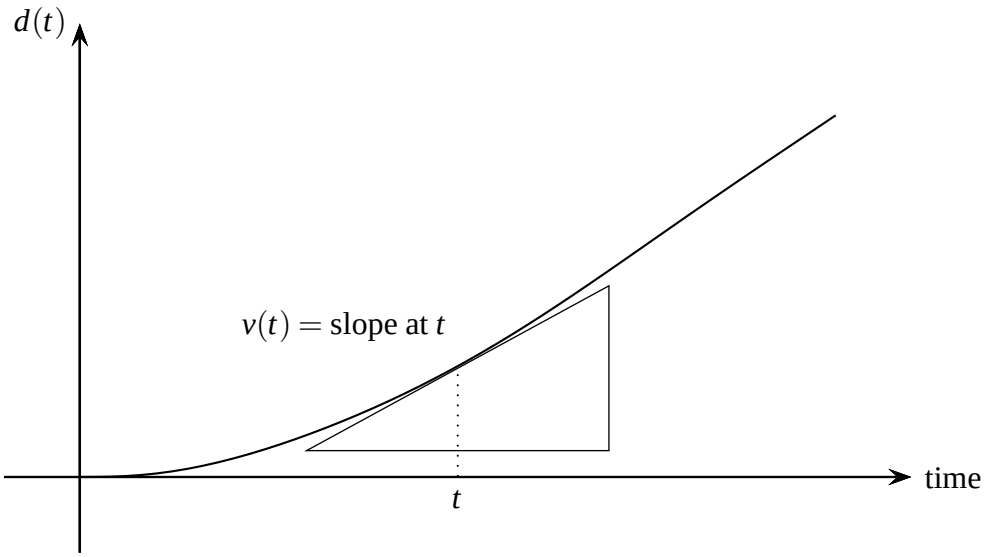


Figure 7.9: The time history of the position, which is determined by computing the area under the velocity curve in Figure 7.8.

The process for obtaining the area under the curve of the velocity profile is the same as obtaining the area from the acceleration curve.

All of these determinations using calculus require that we know the mathematical description of the acceleration curve. However, most of the time we are able to approximate the acceleration as a constant. When we do, the calculus solution gives us the same result as found in Section 7.3.1. Start by determining the velocity from acceleration:

$$\begin{aligned} v(t) &= v_o + \int_{t_o}^{t} a\,ds \\ &= v_o + a\int_{t_o}^{t} ds \\ &= v_o + a\,[s]_{t_o}^{t} \\ &= v_o + a(t - t_o) \end{aligned}$$

If $t_o = 0$, then

$$v(t) = v_o + at \tag{7.17}$$

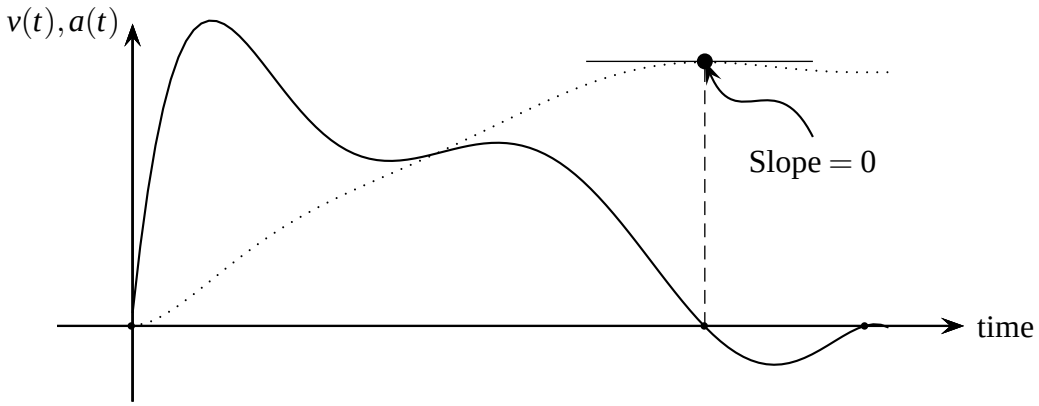


Figure 7.10: The velocity graph (dotted), and the acceleration graph (solid) are superimposed to show how a zero crossing of the acceleration shows a zero slope on the velocity graph because the slope of the velocity curve is the acceleration.

The result, Eq. (7.17), matches the result found in Eq. (7.9) on page 7-10 because we assumed acceleration did not vary with time. We can also write $(\int_{t_o}^{t} a\,ds)$ as $(a\int_{t_o}^{t} ds)$ because a does not vary during the time interval in question.

Continuing with the integration of the velocity from Eq. (7.17) gives a result for displacement under constant acceleration.

$$\begin{aligned} d(t) &= d_o + \int_{t_o}^{t} v(s)\,ds \\ &= d_o + \int_{t_o}^{t} v_o + as\,ds \\ &= d_o + v_o \int_{t_o}^{t} ds + \int_{t_o}^{t} as\,ds \\ &= d_o + v_o\,[s]_{t_o}^{t} + \left[\frac{1}{2}s^2\right]_{t_o}^{t} \\ &= d_o + v_o(t-t_o) + \frac{1}{2}(t^2 - t_o^2) \end{aligned}$$

If $t_o = 0$, then

$$d(t) = d_o + v_o t + \frac{1}{2}at^2 \tag{7.18}$$

Notice this equation matches Eq. (7.12) on page 7-12.

We have just seen how we can go from the acceleration curve to the displacement curve by using

integration. Let's see how we can go the other way from the position curve to the acceleration curve by taking the derivative. Velocity is defined as the time rate of change of displacement and can be written as a derivative:

$$v(t) = \frac{d}{dt}d(t)$$

Since the derivative is the slope at a point, we can visualize the velocity at any time t as the slope of the displacement curve, as shown in Figure 7.9. Some texts will use the dot notation represent derivatives with respect to time, so you may see $v(t) = \dot{d}(t)$.

Likewise, acceleration is defined as the time rate of change of velocity and can also be written as a derivative:

$$a(t) = \frac{d}{dt}v(t)$$

Since we know the velocity is the first derivative of displacement, we can compose the derivative operations and write acceleration as the second derivative of displacement:

$$\begin{aligned} a(t) &= \frac{d}{dt}\left(\frac{d}{dt}d(t)\right) \\ &= \frac{d^2}{dt^2}d(t) \end{aligned}$$

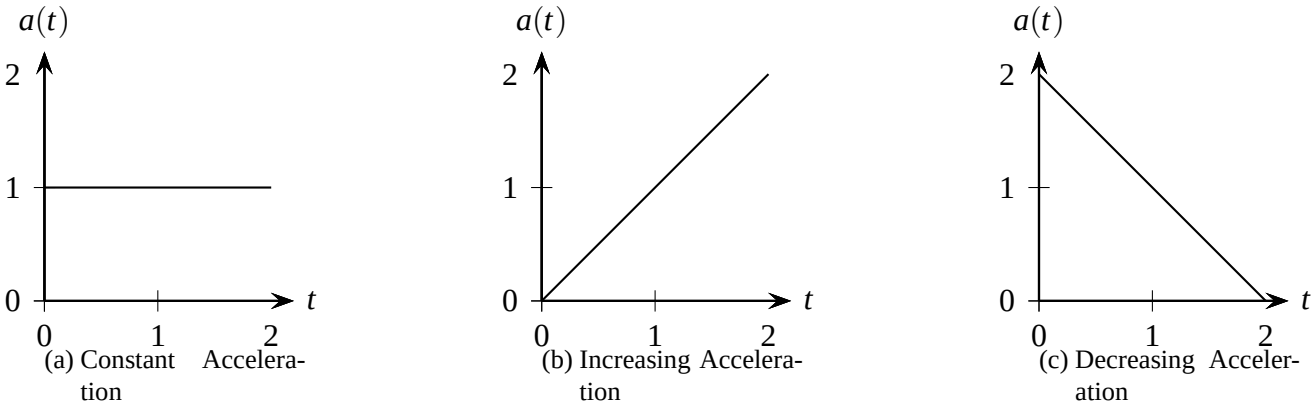
This can be written in dot notation as $a(t) = \ddot{d}(t)$.

Example 7.1. What is the kinematic response of an object over a period of two seconds ($0 \leq t \leq 2$) that has the acceleration profiles defined by:

a.) $a(t) = 1$ (constant acceleration)

b.) $a(t) = t$ (ramp up)

c.) $a(t) = 2 - t$ (ramp down)



(a) Constant Acceleration (b) Increasing Acceleration (c) Decreasing Acceleration

Figure 7.11: The acceleration curves under analysis in Example 7.1.

These acceleration rates are shown graphically in Figure 7.11. The change in velocity corresponds to the area under the acceleration curve. This area is computed using an integral, which is shown in Eq. (7.15). Let's assume the object is initially at rest and its initial point is the origin, $v(t=0)=0$ and $d(t=0)=0$. The areas under the acceleration curves are:

a.) For $a(t)=1$ (constant acceleration),

$$\begin{aligned} v(t) &= v_o + \int_0^t 1\,ds \\ &= 0 + (s|_0^t \\ &= t - 0 \\ &= t \end{aligned}$$

This means that after 2 seconds the velocity is 2, $v(2)=2$.

b.) For $a(t)=t$ (ramp up),

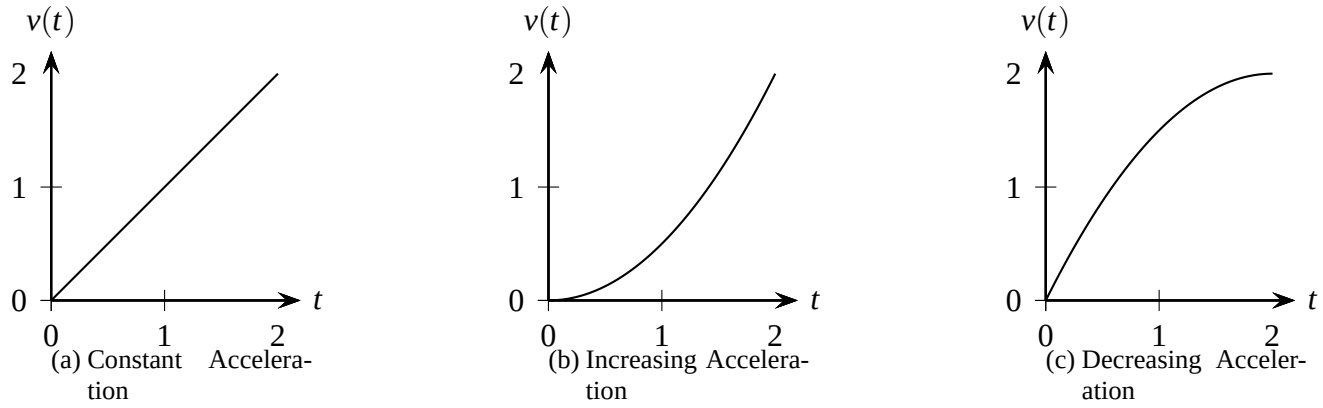
$$\begin{aligned} v(t) &= v_o + \int_0^t s\,ds \\ &= 0 + \left(\frac{1}{2}s^2\right|_0^t \\ &= \frac{1}{2}t^2 - 0 \\ &= \frac{1}{2}t^2 \end{aligned}$$

The velocity at $t=2$ seconds is $v(2)=\frac{1}{2}(2)^2=2$

c.) For $a(t) = 2 - t$ (ramp down),

$$\begin{aligned} v(t) &= v_o + \int_0^t 2 - s\,ds \\ &= 0 + \left(2s - \frac{1}{2}s^2\right)\Big|_0^t \\ &= (2t - \frac{1}{2}t^2) - (0 - 0) \\ &= 2t - \frac{1}{2}t^2 \end{aligned}$$

Again, after 2 seconds the velocity is $v(2) = 2(2) - \frac{1}{2}(2)^2 = 2$.



(a) Constant Acceleration

(b) Increasing Acceleration

(c) Decreasing Acceleration

Figure 7.12: The velocity curves under analysis in Example 7.1.

Notice that in all three cases, the velocities after two seconds are the same! This means the average acceleration for each case is the same. That is, the area under each curve in Figure 7.11 is the same and can be represented as a rectangle with a height of the average acceleration.

Let's see what happens if we try to use the average acceleration to determine the distance traveled for the aforementioned cases. The true displacement is the area under the velocity curves shown in Figure 7.12. It is clear that the area under the curves in Figure 7.12 are not the same. We can show this using the integral:

a.) For $v(t) = t$ (constant acceleration),

$$\begin{aligned} d(t) &= d_o + \int_0^t t\,ds \\ &= 0 + (\frac{1}{2}s^2|_0^t \\ &= \frac{1}{2}t^2 - 0 \\ &= \frac{1}{2}t^2 \end{aligned}$$

This means that after 2 seconds the displacement is 2, $d(2) = 2$.

b.) For $v(t) = \frac{1}{2}t^2$ (ramp up),

$$\begin{aligned} d(t) &= d_o + \int_0^t \frac{1}{2}s^2\,ds \\ &= 0 + (\frac{1}{6}s^3|_0^t \\ &= \frac{1}{6}t^3 - 0 \\ &= \frac{1}{6}t^3 \end{aligned}$$

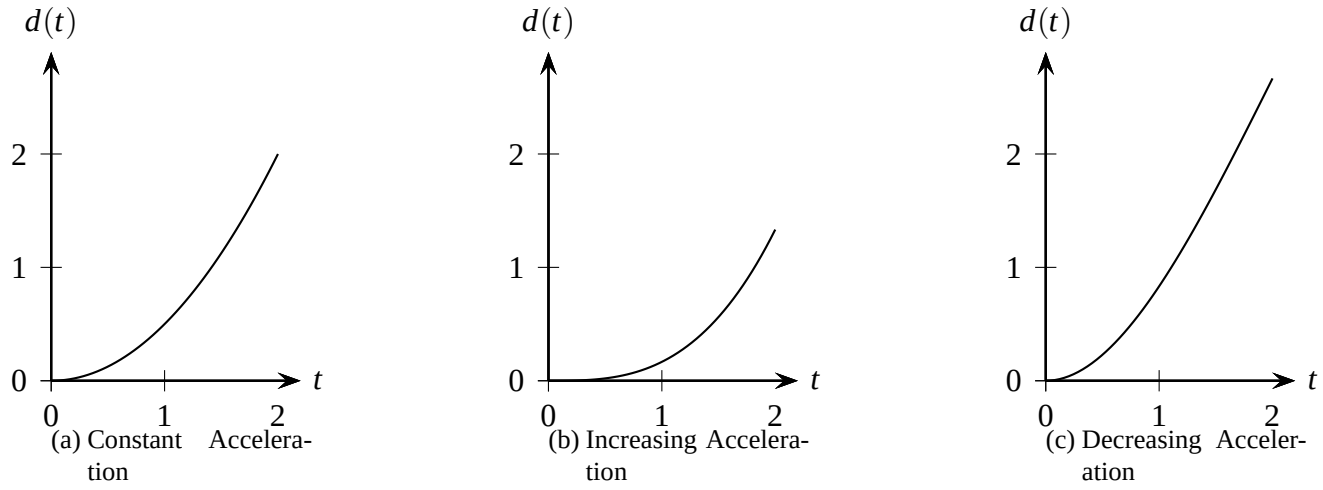
The displacement at $t = 2$ seconds is $d(2) = \frac{1}{6}(2)^3 = \frac{4}{3} \approx 1.3333$

c.) For $v(t) = 2t - \frac{1}{2}t^2$ (ramp down),

$$\begin{aligned} d(t) &= d_o + \int_0^t 2s - \frac{1}{2}s^2\,ds \\ &= 0 + (s^2 - \frac{1}{6}s^3|_0^t \\ &= (t^2 - \frac{1}{6}t^3) - (0 - 0) \\ &= t^2 - \frac{1}{6}t^3 \end{aligned}$$

After 2 seconds the displacement is $d(2) = (2)^2 - \frac{1}{6}(2)^3 = \frac{8}{3} \approx 2.667$.

As we can see from these results (also shown in Figure 7.13), the displacement after 2 seconds is not the same, even though the velocity is! This means we cannot use the average acceleration to determine distance if the acceleration is not uniform. We can, however, use the average velocity to determine the distance traveled. Fortunately, when vehicles brake to maximum, their behavior can be approximated by a constant acceleration (Case a). However, when a car accelerates to speed



(a) Constant Acceleration (b) Increasing Acceleration (c) Decreasing Acceleration

Figure 7.13: The displacement curves under analysis in Example 7.1.

from a stop, the kinematics are approximated best by a decreasing acceleration rate (Case c). This means that if the acceleration is approximated as constant, the distance estimation will be short.

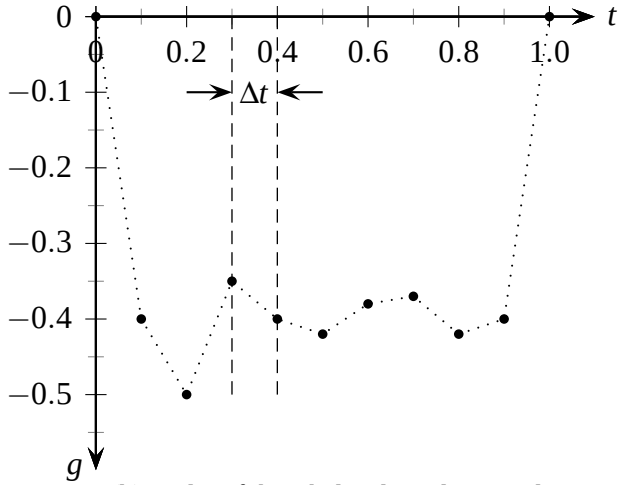
7.4.2 Dealing with Variable Acceleration without Calculus

Understanding the relationship of the area under the acceleration curve to the velocity curve is important because it allows us to simplify the treatment of accelerating objects. Since the velocity at time t is the area under the acceleration curve, we can draw a rectangle that has the same area as the integrated area, and thus the same change in velocity. Since this new area is from a rectangle, the new acceleration curve is now a horizontal line, which means we can use the formulas developed in the previous section for constant acceleration. Further, this means that any time history of acceleration can be approximated as constant acceleration.

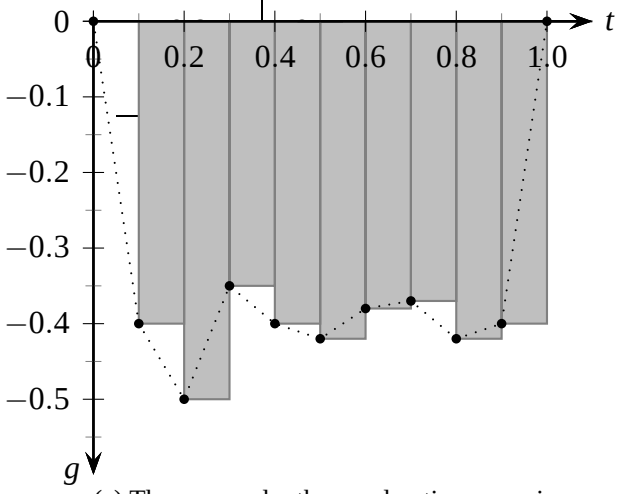
An accelerometer is an instrument that can measure acceleration directly. The instrument will display acceleration values at certain time values. Since those values are rarely the same across consecutive time intervals, we cannot use the formulas in the previous section. Also, there is no mathematical description of the acceleration $a(t)$, so performing the integration is not an option. We do know, however, that the velocity is the area under the acceleration curve, so we can approximate this area by determining the average acceleration on each interval and multiplying it by the width of the time interval. This is the forward Riemann Sum technique of numerical integration. The

Time (S)	Acceleration (g)
0.0	0.00
0.1	–0.40
0.2	–0.50
0.3	–0.35
0.4	–0.40
0.5	–0.42
0.6	–0.38
0.7	–0.37
0.8	–0.42

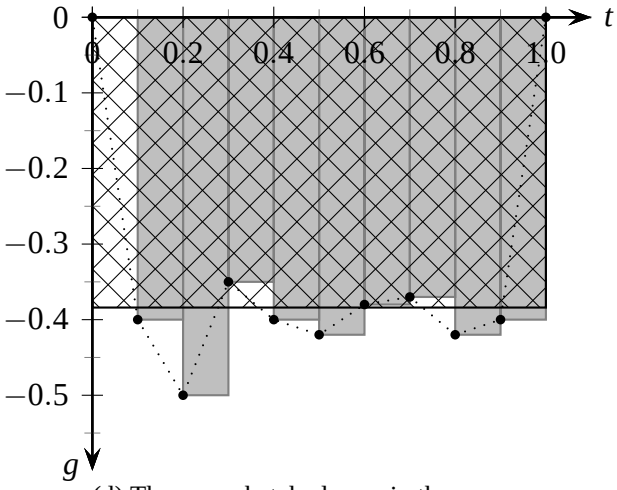
(a) A table of raw acceleration data.



(b) A plot of the tabulated acceleration data.



(c) The area under the acceleration curve is estimated by rectangles at each point.



(d) The cross-hatched area is the same as all the rectangles combined to give an average acceleration.

Figure 7.14: The points measured by a deceleration curve with a dotted line connecting sequential points.

formula for determining the change in velocity using a Riemann Sum rule is:

$$\begin{aligned}\Delta v &= \sum_{i=0}^{n} a_i \Delta t \\ &= a_0\Delta t + a_1\Delta t + a_2\Delta t + \cdots + a_n\Delta t \\ &= \Delta t(a_0 + a_1 + a_2 + \cdots + a_n)\end{aligned} \tag{7.19}$$

In this formula, Δt is the amount of time between samples.[†] Therefore, the total change in time is $t_f - t_0 = n\Delta t$ where n is the number of samples. Now we can divide both sides of Eq. (7.19) by the total change in time, $n\Delta t$:

$$\frac{\Delta v}{n\Delta t} = \frac{\cancel{\Delta t}(a_0 + a_1 + a_2 + \cdots + a_n)}{n\cancel{\Delta t}}$$

The Δt terms cancel on the right and the change in velocity divided by the total time is the definition of the average velocity. So,

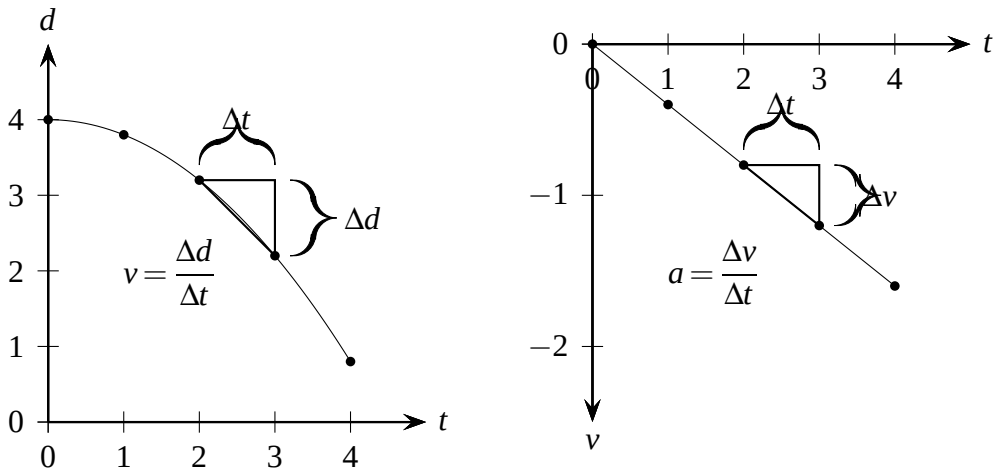
$$\bar{a} = \frac{(a_0 + a_1 + a_2 + \cdots + a_n)}{n}$$

which is simply the average of all the data points making up an acceleration measurement. This means that there is an equivalent constant acceleration that will give the same velocity change as for a variable acceleration. This allows us to use the constant acceleration formulas to analyze events that happen over some period of time. If the acceleration has a lot of jerk, then the time intervals need to be broken up into smaller pieces, for which a constant acceleration assumption is more applicable.

The exact same procedure is used to obtain a change in displacement when we have samples of velocity. Problems like these can become large if there are a lot of data points. The use of computers and spreadsheets greatly enhance our ability to solve problems when acceleration is variable.

What about going the other way? What if we have data points that are positions at specific time intervals? We can only obtain average velocities and accelerations from these by taking the slope between successive points.

[†] If Δt becomes very small, then we have the definition of an integral from Section **??** on page ??.



(a) Displacement points

(b) Velocity points from the different slopes in (a).

Figure 7.15: Illustration of going from displacement to velocity. Since velocity follows a straight line, the acceleration is constant.

Say we are given a series of displacements at fixed intervals in time, as shown in Figure 7.15a. The velocity at any given time is the slope between each point. As we determine velocity from displacement, we can only get an average velocity over the interval. An example of this is when traffic engineers set up two equally spaced sensors across the road to measure an average (constant) speed. This is done by dividing the distance between the sensors by the time between activation of each sensor. The reason we put "constant" in parentheses is because the average velocity would be the velocity the object would travel if it had a constant speed. This also means that whether a car started from a stop or was cruising at the same speed, and it traveled some distance in the same amount of time, then its average velocity would be the same. As a result, we cannot tell if an object is accelerating if only two points in time are known.

So how do we know if an object is accelerating? We either have to know the speed at the beginning, or better yet, break up the distance and times into smaller increments. This way, there can be multiple determinations of velocity and thus multiple determinations of acceleration. This is important for variable acceleration motion, such as a vehicle accelerating at full throttle.

If we read some automobile magazines, we will find that vehicle performance is reported as the time taken to go from 0-30 mph, 0-60 mph, and sometimes to the quarter mile. These times give different accelerations because of the difference in the power bands of the engines, and because of the variable effects, like wind.

Acceleration data determined from speeds or distances must be used only for the same types of

situations. We would never compare the average acceleration over a quarter mile to the acceleration of a vehicle pulling out from a stop sign because the distance intervals are too different.

The main point of this discussion is to understand the implications of determining velocity and acceleration information from time–distance determinations. To gain more accurate information, the distance should be smaller. Smaller distances result in shorter times. If the time period is short, then the time measuring device must be very accurate because we would be dividing by a small number. Whenever we divide by a relatively small number, variations on that number can potentially have a large effect on the final answer. This is known as *sensitivity* and we would say that for small time intervals, the velocity determination is sensitive to the time measurements.

On the other hand, determining the velocity and displacement from acceleration data is very stable and smooth because we are averaging at each step. Given a choice, it is always better to determine values by averaging rather than by differencing.

7.5 Relative Motion

When there are two objects moving in a straight line together, the motion of one object can be described from the other. This perceived motion is easily understood when driving on a four-lane highway and being passed by someone “slowly.” Even though your ground speed is high, the relative speed of the other vehicle is low. *Closing speed* is also defined as the relative velocity of one object to another. Here are some definitions:

Reference frame is a coordinate system from which relative or absolute measurements or observations can be made. Those measurements become dependent on the reference frame.

Absolute motion is motion when referenced to an earth-fixed coordinate system. The origin of an absolute or global coordinate system does not change with respect to the earth. These are also known as inertial reference frames, which means they will never accelerate or rotate.

Relative motion describes the motion of an object as referenced from two different moving reference frames. When a person in a vehicle moves relative to the vehicle, we use relative motion to describe occupant kinematics.

Closing speed is the magnitude of the difference of the velocity vectors of two objects.

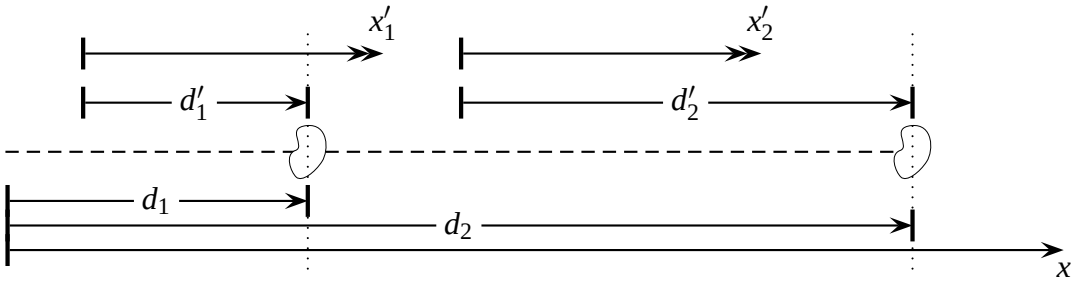


Figure 7.16: Displacement of a particle as observed by a fixed reference frame, x, and a moving reference frame, x'. The subscripts denote different snapshots in time. Notice the displacements d' are different than d, even though they describe the displacement of the same object.

When we defined rectilinear motion in Section 7.2, we defined it to be one dimensional in the positive x-direction. However, determining relative motion in one dimension requires us to think of vectors. The sign of the vector determines direction. The relative quantity of motion is the difference between the motion of one object and the motion of the second object. This may be best illustrated with an example.

7.5.1 Example

A pickup and a car are traveling toward each other over a bridge. Consider the left end of the bridge as the origin and the positive x-axis extends down the bridge to the right. The truck is 50 ft (15.24 m) from the end of the bridge and the car is 125 ft (38.1 m) from the end of the bridge. What is the relative displacement of the truck from the car?

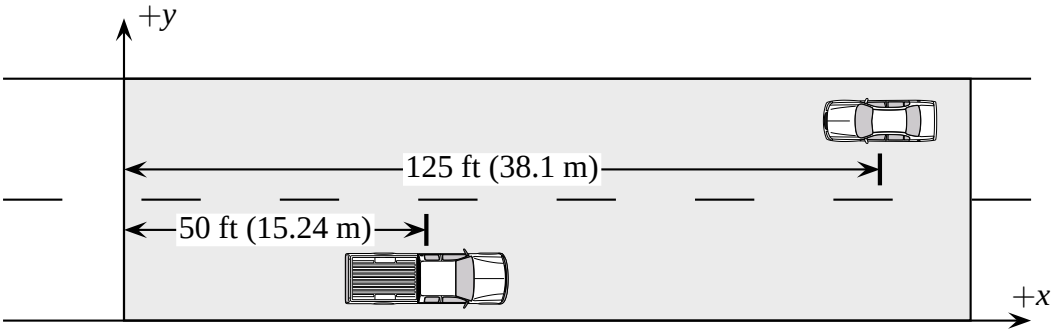


Figure 7.17: A diagram to accompany Example 7.5.1.

The answer is the difference of the two values, but we need to know which number to subtract from the other. Obviously, both displacements are positive and the distance between them is 75 feet (22.86 m), but displacement needs to have a direction, so is it +75 ft (+22.86 m) or –75 ft (–22.86

m)? Since we asked about the relative motion of the truck in relation to the car, the displacement of the car is subtracted from the displacement of the truck, so the answer is –75 ft (–22.86 m). This means that an observer would have to travel 75 ft (22.86 m) in the negative direction to reach the truck.

The next question is what is the closing speed between the two vehicles if the car is traveling 50 mph (80 kph) and the truck is travelling 30 mph (48 kph)? To convert these speeds to velocities, we need to know the directions of travel. This gives the velocity of the car as –50 mph (–80 kph) and of the truck as +30 mph (+48 kph). The closing speed is just the magnitude of the difference, which is 80 mph (128 kph). If the relative velocity of the car with respect to the truck is desired, the velocity of the truck is subtracted from the velocity of the car, so $-50-30=-80$ mph ($-80-48=-128$ kph). Similarly, the relative velocity of the truck with respect to the car is determined by subtracting the velocity of the car from the velocity of the truck, i.e. $30-(-50)=80$ mph ($48-(-80)=128$ kph).

Exercises

1. If an object is traveling at a constant velocity of -30 ft/s,

   a) What distance will be traveled in 5 seconds? $\boxed{150\,\text{ft}}$

   b) At time $t = 3$ seconds, the object is at the origin ($x = 0$), what is its displacement at time $t = 0$? $\boxed{90\,\text{ft}}$

   c) Again, at time $t = 3$ seconds, the object is at the origin ($x = 0$), what is its displacement at time $t = 5$ seconds? $\boxed{-60\,\text{ft}}$

   d) How long would it take for the object to be 180 ft from the origin? $\boxed{6\,\text{sec}}$

2. A car accelerates at a constant rate of 0.2 times the acceleration due to gravity, $g = 32.2$ ft/s/s, for 2.5 seconds.

   a) How much change in velocity does the vehicle experience? $\boxed{16.1\,\text{ft/s}}$

   b) If the vehicle started from a stop, how far would it have traveled? $\boxed{20.13\,\text{ft}}$

   c) If the vehicle started from a stop, how fast would it be going? $\boxed{10.98\,\text{mph}}$

   d) If the vehicle had some initial velocity, would it cover the same amount of distance as if it were starting from a stop? $\boxed{\text{no}}$

3. A car skids to a stop. The acceleration of a car is reported in discrete times as shown below: (Hint: Use trapezoid areas)

time (sec)	0.0	0.4	0.8	1.2	1.6	2.0	2.4
acceleration (m/s^2)	-2.0	-7.0	-6.0	-5.5	-6.0	-6.0	0.0

   a) What was the change in velocity over the interval $t = [0.0, 0.4]$? $\boxed{-1.8\,\text{m/s}}$

   b) How fast was the car originally going? $\boxed{12.6\,\text{m/s}}$

   c) What is the average acceleration of the skid? $\boxed{-5.25\,\text{m/s}^2}$

   d) How far did the vehicle skid? $\boxed{14.4\,\text{m}}$

   e) If we assume the acceleration was constant and had the value from question c.), how much distance would have been traveled? Why is there a difference? $\boxed{15.12\,\text{m}}$

4. A camera is set up to take a picture every half second. We have three frames that show a particular car as it drives down a road. From the interpretation of the photographic evidence,

it was determined that the car traveled 20 ft in the first interval and 18 ft in the second interval. The total distance of the car is reported at discrete times, as shown below:

time (sec)	0.00	0.50	1.00
displacement (ft)	0	20	38

40 ft/s a) What is the average velocity in the first interval?

36 ft/s b) What is the average velocity in the second interval?

$8\,\text{ft/s}^2$ c) What is the average acceleration?

What if we learned later that the timer was off by 0.03 seconds so the new data table appears as:

time (sec)	0.00	0.47	0.94
displacement (ft)	0	20	38

42.55 ft/s a) What is the average velocity in the first interval?

38.29 ft/s b) What is the average velocity in the second interval?

$9.06\,\text{ft/s}^2$ c) What is the average acceleration?

-6% d) What is the relative change in the timing?

13.25% e) What is the relative change in the acceleration?

f) Why does the error in timing make a larger error in acceleration?

5. If two vehicles are traveling down the road at 25 m/s and vehicle #1 applies the brakes and slows at a constant rate of $-2\ \text{m/s}^2$ while vehicle #2 maintains current velocity,

12.5 sec a) How much time does it take for vehicle #1 to come to a stop?

156.25 m b) How far is vehicle #2 from vehicle #1 when vehicle #1 comes to a stop?

4 m/s c) What is the relative speed between the two vehicles 2 seconds after brake application?

8 Motion in More than One Dimension

8.1 Degrees of Freedom

A degree of freedom may be thought of as motion along or about an axis in a system like the one illustrated in Figure 8.1 on page 8-2. This system is a 3-dimensional system with 3 axes: *x*, *y*, and *z*. Though a car may flex a little, we consider it to be a rigid body. A rigid body can move in a combination of six different ways in space and is said to have six degrees of freedom. In other words, a car may travel (translate) along any axis in either direction (3 degrees of freedom) or rotate about any axis in either direction (the other degrees of freedom). The car may also travel along an axis and at the same time rotate about that axis. A human elbow joint, on the other hand, would have only one degree of freedom, the motion of the forearm about a single axis.

We do not always use all degrees of freedom in our calculations. Doing so is often not necessary. We should also be aware that articulated* vehicles, when viewed as a system, may have more than six degrees of freedom.

8.2 Motion in Space

When discussing a vehicle, the coordinate system associated with the vehicle is generally referred to as a *vehicle-fixed* coordinate system. These coordinates are normally written in lower case letters.

The coordinate system associated with the vehicle environment is generally referred to as an *earth-fixed* coordinate system or a *global* coordinate system, with the respective coordinates referred to as earth-fixed or global coordinates. These coordinates are usually written with capital letters, as shown in Figure 8.3.

*An articulated vehicle has hinged joints which allow rotational motion between two or more parts of the vehicle. Motorcycles, tractor-trailers, and trains are all articulated vehicles.

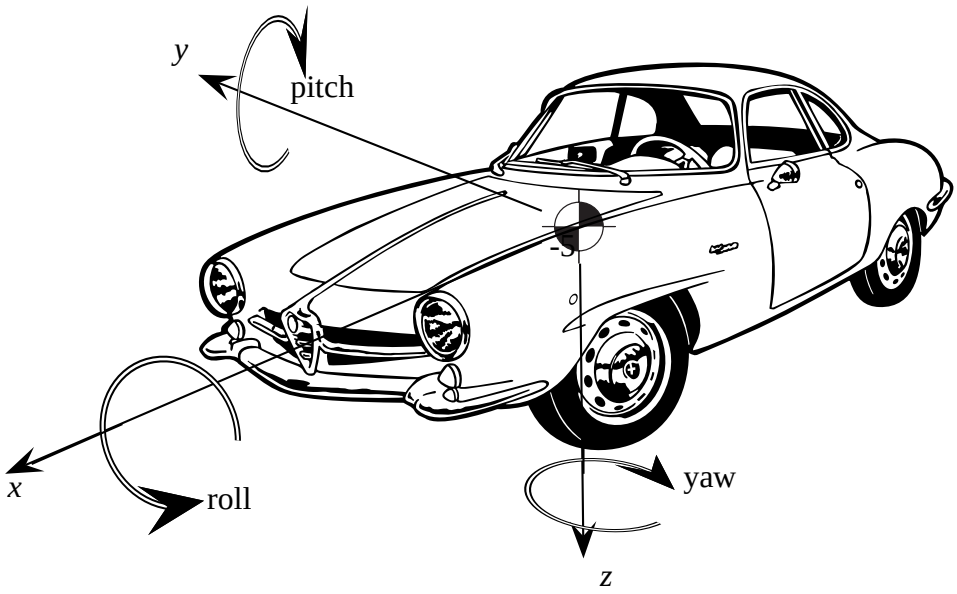


Figure 8.1: The typical six degrees of freedom of a car. All directions are shown as positive according to SAE standards [3].

As an object in motion moves through space, its path may include components of all three dimensions (x,y,z) in varying amounts during any given moment. It may also have pitch, yaw, and roll motion, which are defined as rotation about the y, z, and x axes, respectively. Consequently, the path the object travels may not be a straight line. If the (x,y,z) axes of the object remain parallel to the (X,Y,Z) axes of the environment, then the object is said to be undergoing purely translational motion. In this case, every particle in the object undergoes the same displacement as every other particle in the object. For example, the object shown in Figure 8.2 is undergoing translational motion as it moves along its path since its axes remain parallel with the environmental axes. If the (x,y,z) axes of the object do not remain in the same alignment with the (X,Y,Z) axes of the environment, then the object is rotating.

For the remainder of this chapter we will be discussing rigid body translation in a plane without rotation. This means there will be two coordinates used to describe the displacement, velocity, and acceleration vectors. Most analysis we do as reconstructionists involve only two dimensions because the effect of the third dimension will not change the results of the 2-D case. However, if need be, the extension into three dimensions is not difficult once we gain an understanding of planar motion.

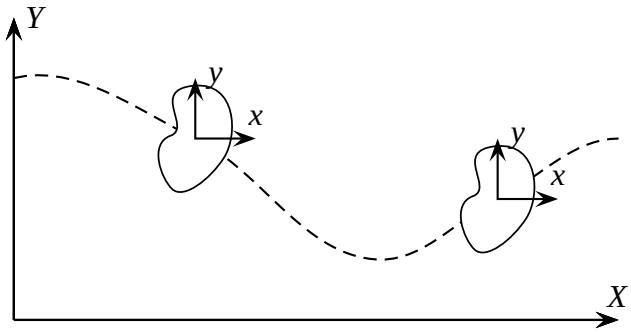


Figure 8.2: An object undergoing only translation along a curved path. The local object coordinates (x,y) remain aligned with the environmental global coordinates (X,Y).

Often, a full-blown 3-D reconstruction can be broken down into smaller problems only in 2-D. For example, if a car comes around a corner really fast then loses control and goes off a bridge, then there are definitely three dimensions to consider. But, we can break the problem up into two separate planar events: 1) the time while the car is on the road and 2) the time the car is airborne. These individual events each occur on a plane (or near plane). Now that we understand the importance of planar motion, let us begin. . .

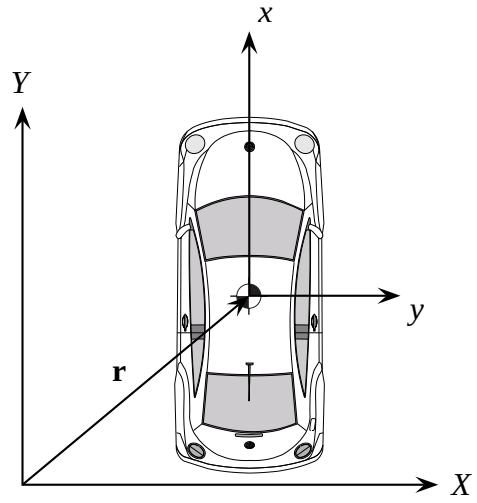


Figure 8.3: An illustration of earth-fixed and vehicle fixed coordinate systems. Notice that the vehicle axes correspond to the SAE standard.

8.3 Displacement

The position of a particle is defined by its coordinates. Those coordinates represent a vector based on some predefined coordinate system or basis. Let's assume we have a Cartesian (x,y) coordinate

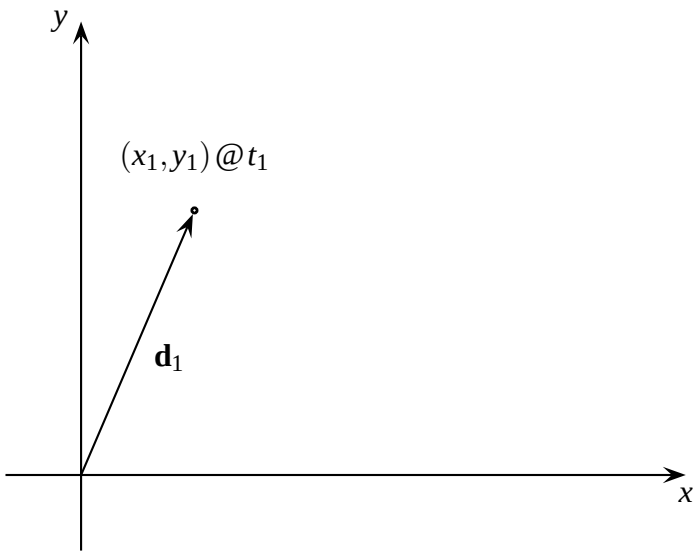


Figure 8.4: The position of a particle in an (x,y) coordinate system at time t_1. The vector $\mathbf{d_1}$ is used to represent the coordinates (x_1, y_1).

system and the object we are analyzing is at the position (x_1, y_1), as shown in Figure 8.4. Physics allows us to treat any arbitrary object as a point, located at the center of mass, to study translational motion.

Now the particle in Figure 8.4 is going to experience motion and be displaced to a new point. This displacement takes some finite time and the path to get there is arbitrary. The object is at a new position at time t_2 denoted by the vector $\mathbf{d_2}$ with coordinates (x_2, y_2), as shown in Figure 8.5.

8.4 Speed and Velocity

8.4.1 Average Velocity

We know that the velocity of an object is defined as the rate at which the position of the object changes with respect to time. By recalling Figure 8.5, we see that there is a finite time for the object to move from $\mathbf{d_1}$ to $\mathbf{d_2}$. This time is simply $\Delta t = t_2 - t_1$.

The displacement vector describing the change in position of the vehicle as it travels from $\mathbf{d_1}$ to $\mathbf{d_2}$ is $\Delta\mathbf{d}$, which is equal to $\mathbf{d_2} - \mathbf{d_1}$. Thus

$$\mathbf{v}_{avg} = \frac{\mathbf{d_2} - \mathbf{d_1}}{t_2 - t_1} = \frac{\Delta \mathbf{d}}{\Delta t} \tag{8.1}$$

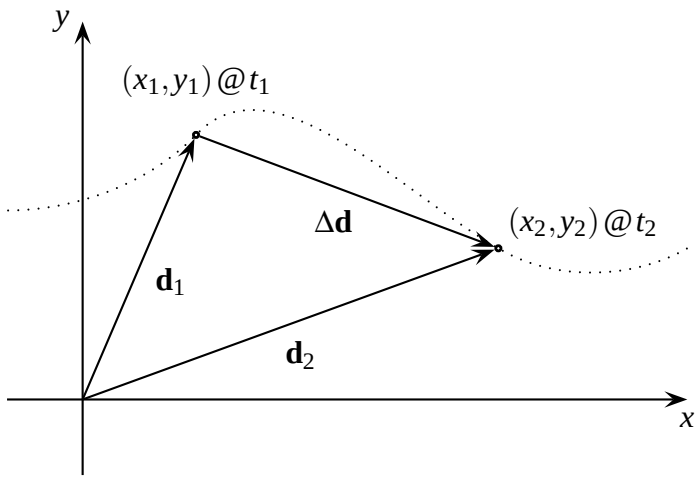


Figure 8.5: The new position of a particle time t_2 after traveling an arbitrary path. The change in displacement is the vector $\Delta\mathbf{d}$ and is the difference between $\mathbf{d_2}$ and $\mathbf{d_1}$.

Now keep in mind that we have defined an average velocity vector that has both magnitude and direction. To determine the magnitude of the average velocity vector, we must find its length. But in order to determine the length, we must know the components of the vector. Recall that we can break up the velocity vector into components: $\mathbf{v}_{avg} = (v_{avg,x}, v_{avg,y})$. Similarly, the displacement vector is broken up as $\mathbf{d} = (x, y)$. To determine the value of each component, perform the calculation defined by Eq. 8.1 on each component:

x component	y component
$v_{avg,x} = \dfrac{x_2 - x_1}{t_2 - t_1} = \dfrac{\Delta x}{\Delta t}$	$v_{avg,y} = \dfrac{y_2 - y_1}{t_2 - t_1} = \dfrac{\Delta y}{\Delta t}$

With both components of the average velocity vector known, its magnitude can be found using Eq. (6.3) found on page 6-7:

$$v_{avg} = |\mathbf{v}_{avg}| = \sqrt{(v_{avg,x})^2 + (v_{avg,y})^2} \tag{8.2}$$

Equation (8.2) defines the average velocity of the vehicle as it travels from $\mathbf{d_1}$ to $\mathbf{d_2}$ since it involves only the total displacement the vehicle undergoes and the total time it takes to do it. Recall the traveler who drives from Springfield, Illinois to Jackson, Wyoming and back again, in Figure 7.1.

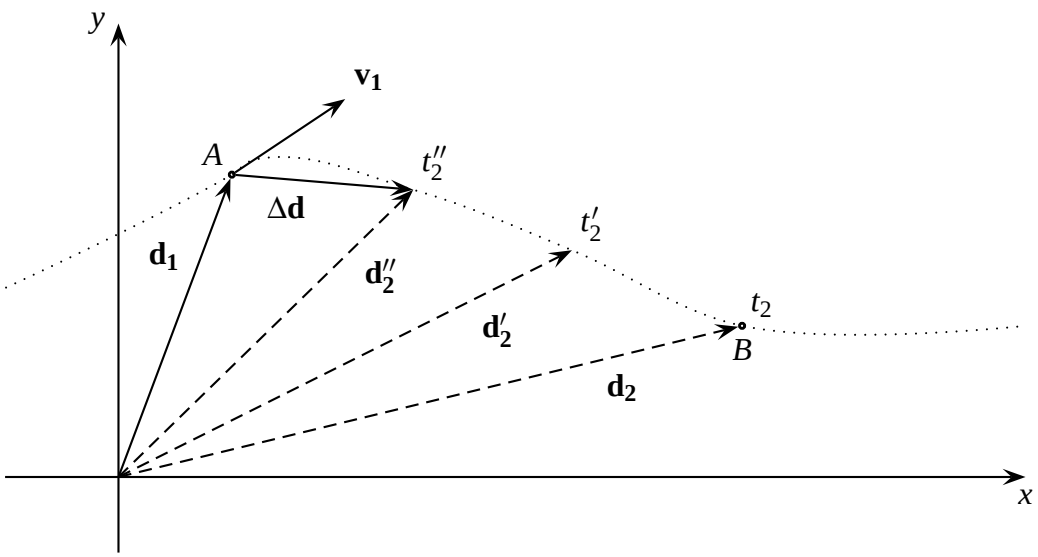


Figure 8.6: As the time interval becomes shorter, the displacement of an object becomes smaller. In the mathematical process of taking the limit, the instantaneous velocity vector is defined.

In time Δt, the traveler has an average velocity of zero for the trip because the traveler's total displacement is also zero for this time interval.

8.4.2 Instantaneous Velocity

As the object moves from *A* to *B* in Figure 8.6, it does not move with constant velocity. Although the average velocity of the object is defined by Eq. (8.1), nothing has been said or determined about the changing velocity of the object at any given point along the path. That is, nothing has been said about the instantaneous velocity at any given moment in time. Consider once more the example of the traveler who drives from Springfield, Illinois to Jackson, Wyoming and back again in time Δt, with an average velocity of zero. This traveler nevertheless has some non-zero value of instantaneous velocity at any given point in time while traveling.

So, suppose we wanted to know the velocity of the object at point *A* in Figure 8.6. The velocity vector at point *A* is $\mathbf{v_1}$ and is tangent to the curve at point *A*. If we can determine the slope of the velocity vector, we will have determined the rate of change of the vector, and thus the value of the velocity at point *A*.

Therefore, if we allow point *B* to get closer and closer to point *A*, the slope of the displacement vector $\Delta\mathbf{d}$ gets closer and closer to the slope of vector $\mathbf{v_1}$.

Moving the location of B closer and closer to A can be accomplished by making Δt smaller and smaller.

Thus, we are taking a limit, as defined in Section **??**:

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\blacksquare\mathbf{d}}{\Delta t} \tag{8.3}$$

In calculus this is taking the derivative of $\mathbf{d}$ with respect to t:

$$\mathbf{v} = \frac{d}{dt}(\mathbf{d}(t)) \tag{8.4}$$

where $\mathbf{d}(t)$ is the displacement vector as a function of time, and the magnitude of the instantaneous velocity vector is the speed (a scalar quantity).

8.5 Acceleration

8.5.1 Average Acceleration

Acceleration may be defined as the rate of change of velocity over a period of time. In other words, acceleration is the measurement of how fast the magnitude or direction, or both, of the velocity of an object changes. A change in magnitude can result either in positive or negative acceleration. If the acceleration is positive, the object will speed up. If the acceleration is negative, the object will slow down. Acceleration will also result if the direction component of the velocity changes. Since velocity is a vector, a change in velocity can be accomplished by changing either its magnitude or its direction, or both.

Suppose we again look at the vehicle traveling from A to B as shown in Figure 8.7 on page 8-8. At time t_1 the vehicle is at point A and has an instantaneous velocity of $\mathbf{v_1}$. At time t_2, the vehicle is at point B and has an instantaneous velocity of $\mathbf{v_2}$. The change in velocity is therefore equal to $\mathbf{v_2} - \mathbf{v_1}$ during the time it takes for the vehicle to go from A to B. The time is equal to $t_2 - t_1$. Thus:

$$\mathbf{a}_{avg} = \bar{\mathbf{a}} = \frac{\mathbf{v_2} - \mathbf{v_1}}{t_2 - t_1} = \frac{\blacksquare\mathbf{v}}{\Delta t} \tag{8.5}$$

In a manner similar to that in which the average velocity equation defines average velocity, this equation defines the average acceleration of the vehicle as it travels from A to B since it involves

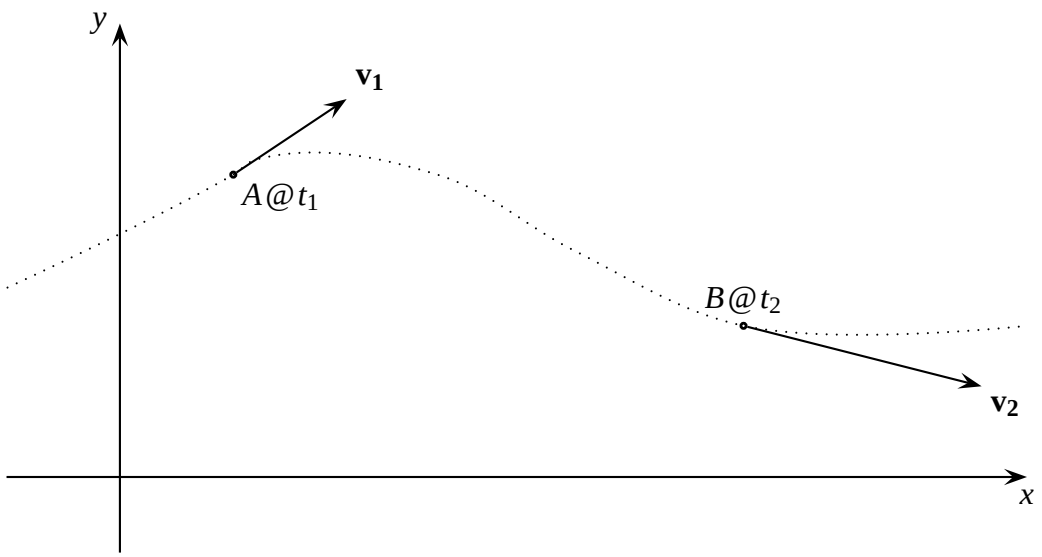


Figure 8.7: Schematic of a change in velocity over some change in time.

only the total change in velocity the vehicle undergoes and the total time it takes to change. Additionally, as in average velocity, nothing has been said or determined about the changing acceleration of the vehicle at any given point along the path. In other words, nothing has been said about the instantaneous acceleration at any moment in time.

8.5.2 Instantaneous Acceleration

Finding the instantaneous acceleration of the object in Figure 8.7 follows the same procedure of taking the limit, as shown in Eq. (8.3). However, as mentioned in Section 7.4, the change in velocity due to a variable acceleration can be approximated by an equivalent constant acceleration. The same principle holds for motion in more than one dimension.

8.6 Uniform Projectile Motion

When we examine the displacement, velocity, and acceleration of objects or vehicles on the ground, the movement is generally along a straight-line path (usually the x-axis) and is one dimensional motion (rectilinear motion). When objects, or vehicles, move through the air, they are moving through a plane and are moving in two dimensions. Thrown or struck baseballs, thrown or kicked footballs, golf balls, diving swimmers, or vehicles leaving an embankment are all examples of objects in projectile motion. This motion is taking place in two dimensions. In examining projectile

motion it will be assumed that the effect of air resistance is negligible and can be ignored and that the acceleration of gravity is constant with a magnitude of 32.2 fps^2 (9.81 m/s^2) and is directed downward.

We may have heard the physics class example of a gun firing a bullet vs. a dropped bullet. In this example, we have a gun that has its barrel parallel to a perfectly level field. The gun is fired and as soon as the bullet leaves the barrel of the gun, another unfired bullet is dropped to the ground from the same height as the barrel. The question posed to the students is which bullet will strike the ground first? Interestingly, both bullets would hit the ground at the same time assuming no aerodynamic effects. How is this so?

Galileo was the first to analyze and accurately describe projectile motion. He did this by analyzing the horizontal and vertical motions of a projectile separately. If we do the same and examine the horizontal and vertical motion of both bullets separately, we will see why both will hit the ground at the same time.

In the vertical direction, the instant just before the fired bullet leaves the gun, it has no vertical movement (the barrel is perfectly level). The same is true for the dropped bullet as it was being held stationary. However, the instant the fired bullet leaves the gun it begins to fall, being acted upon solely by gravity. The same is true for the dropped bullet the instant it is released. Thus, in the vertical direction, since both bullets are being acted upon only by the force of gravity and since both bullets are falling the same vertical distance and since gravity is acting on both bullets equally, they must take the same amount of time to fall the distance to the ground. Hence, they will hit the ground at the same time.

In the horizontal direction, the fired bullet obviously had a large component of horizontal velocity. On the other hand, the dropped bullet had no horizontal motion. Thus, although both bullets had vastly different horizontal motions, they still fell to the ground in the same amount of time. The horizontal motion of each bullet had no effect on its vertical motion. This illustrates that the motion of a body in one direction is independent of the motion of that same body in another direction.

If we were to look at the trajectories of both bullets, they would look like Figure 8.8. By considering the horizontal and vertical velocity vectors of the fired bullet separately, mathematical expressions can be developed for each of them.

When the bullet is fired in the horizontal direction, it is accelerated through the barrel. Once the bullet leaves the barrel, the acceleration phase is over and the bullet now travels with constant velocity. We learned in Section 7.2.1 on page 7-4 about objects traveling at a constant speed. The horizontal motion of the fired bullet, at the moment it leaves the barrel, can be expressed by the

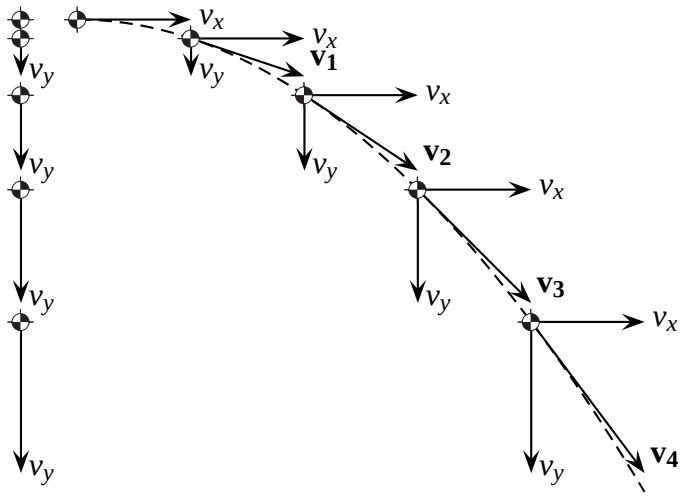


Figure 8.8: Two differing trajectories of a bullet. The dropped bullet is on the left and has no horizontal component. The fired bullet is on the right and has a constant horizontal velocity v_x. The instantaneous velocity vector $\mathbf{v}$ is broken into x and y components with magnitudes v_x and v_y, respectively. Both bullets accelerate toward the ground at the same rate and have the same magnitude for their vertical velocity component.

equation:

$$d = vt \tag{8.6}$$

where d is the horizontal distance traveled, v is the average velocity of the body, and t is the time.

The fired bullet has no vertical movement while it is traveling through the barrel. The instant the bullet leaves the gun it begins falling, or accelerating downward, due to gravity. We recall from Section 7.3.2 on page 7-10 that the equation for the distance traveled by an object that is accelerating is:

$$d = \frac{1}{2}at^2 \tag{8.7}$$

where d is the distance traveled, a is the acceleration, and t is the time.

Since the fired bullet is a falling body, we will modify the symbols for distance and acceleration. Acceleration now becomes the acceleration of gravity $a = g$. Additionally, since distance now becomes the height from which the object falls, the variable h can be substituted for d in Eq. (8.7). Thus, the equation now looks like this:

$$h = \frac{1}{2}gt^2 \tag{8.8}$$

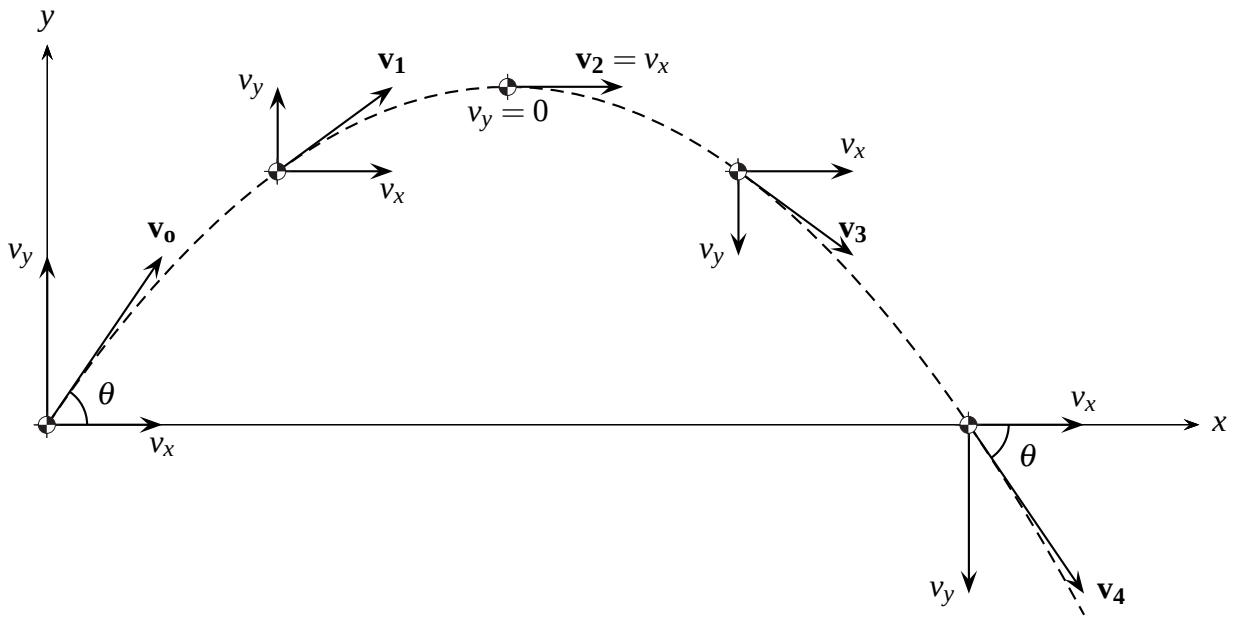


Figure 8.9: Path of an object projected at an upward angle θ with an initial velocity $\mathbf{v_o}$. The path is a parabola and the object maintains a constant horizontal velocity v_x. The peak of the trajectory occurs when the vertical velocity is zero $v_y = 0$.

By examining the time required for the fired bullet to travel through its vertical distance as opposed to the time required for it to travel through its horizontal distance, we realize that the time is identical for both distances. The time required for the bullet to fall is necessarily the same as the time available for it to traverse the distance from where it was fired to where it lands. In other words, the horizontal distance the fired bullet travels must be accomplished during the time that it takes to fall to the ground. Therefore, we can express time in terms of one of its variables by rewriting Eq. (8.6) as:

$$t = \frac{d}{v} \tag{8.9}$$

Substituting Eq. (8.9) for t in Eq. (8.8) yields:

$$h = \frac{1}{2} g \left(\frac{d}{v} \right)^2 \tag{8.10}$$

where h is the falling height, g is the acceleration of gravity, and d is the distance traveled horizontally.

If we look at a vehicle instead of a fired bullet, traveling at some unknown velocity, leaving a level surface, and becoming airborne, the reasoning does not change. We can utilize Eq. (8.10) to determine the velocity of the vehicle as it left the surface. In Eq. (8.10) there are two things we can

measure at a crash scene, namely, d and h. It should be noted that height h will always be measured as the net change in vertical distance of the center of mass of the vehicle from the takeoff point to the point of first touch. Similarly, distance d will always be measured as the change in horizontal distance of the center of mass of the vehicle from the takeoff point to the point of first touch. Since we know the value for g, we can calculate v, which is the value we want.

If an object (e.g., a vehicle) becomes airborne at an angle rather than level, we can use a similar analysis. The only difference is that the object will now have an initial component of vertical velocity as well as an initial horizontal component of velocity. The use of trigonometry is required to make the necessary adjustments for angled takeoffs. The logic will be the same, however, and the resulting equation will be very simple. We still ignore the influence of air resistance in our analysis since the air resistance is negligible for the low speeds and relatively short fall distances we encounter as crash investigators.

Figure 8.9 on page 8-11 represents the trajectory of a object that is projected at an upward angle. In Figure 8.9, the object becomes airborne at an angle of θ from the horizontal and is traveling at an initial velocity of $\mathbf{v_0}$. Since the object is traveling at an angle when it becomes airborne, it has a horizontal and a vertical velocity component, v_x and v_y, respectively. Gravity is continually acting on the object the moment it becomes airborne, so the object will slow, in the vertical direction, until it comes to a stop (reaching the peak of its trajectory), then begin to fall back to the ground. The path of the object is a parabola, which means that the path of the object from takeoff to the peak is symmetrical to the path of the object from peak to the level of takeoff. The symmetry of the path also indicates that the magnitude of the velocity of the object at takeoff is equal to the magnitude of the velocity of the object when it crosses the level of the takeoff. The direction of the velocity vector as it crosses the plane of the takeoff is at an angle of θ below the horizontal. The vertical velocity component will have the same magnitude as the vertical velocity component of the object at takeoff, but in the opposite direction. The magnitude of the horizontal velocity component remains constant throughout the flight of the object.

The important point about uniform projectile motion is that the horizontal and vertical motions are independent of one another. The only way we can couple the equations for horizontal and vertical motion is through time.

Exercises

Exercise 8.1. A car is turning across an intersection and is filmed by a red light camera. The left front tire is seen rolling over a manhole cover when the clock reads 13:52:05.25 (HH:MM:SS). The coordinates (in feet) of the center of the manhole cover are (5,25). Then, when the clock reads 13:52:06.75, the left front tire is rolling over the corner of the stop bar. The coordinates of the corner of the stop bar are (20,0).

a.) What is its average speed? $\boxed{13.25\,\text{mph}}$

b.) Why did we not ask about acceleration?

Exercise 8.2. Develop an equation for initial velocity (ft/s, m/s) using Eq. (8.10).

Exercise 8.3. A particle has a constant velocity in the x-direction of 30 m/s and is falling due to gravity. Neglecting air resistance, how much time will it take for the particle to fall 10 meters, if

a.) there is no initial vertical velocity? $\boxed{1.42\,\text{sec}}$

b.) the vertical velocity is initially 3 m/s up? (i.e., What is the total time of flight if the particle rises to a peak and lands 10 meters below takeoff?) $\boxed{1.76\,\text{sec}}$

c.) the takeoff angle is 340° (-20°) and the horizontal component of the velocity is still 30 m/s? $\boxed{0.69\,\text{sec}}$

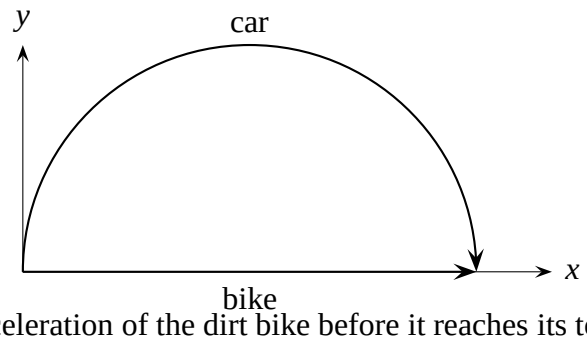
Exercise 8.4. An east-bound pickup traveling at 60 mph hits a north-bound station wagon traveling at 30 mph in the center of an intersection. Let east be the x-axis.

a.) How far was the station wagon from the point of impact 3 seconds prior to impact? $\boxed{-132\,\text{ft}}$

b.) How far was the pickup from the point of impact 3 seconds prior to impact? $\boxed{-264\,\text{ft}}$

c.) What is the closing velocity between the two vehicles immediately prior to impact? $\boxed{67\,\text{mph}}$

Exercise 8.5. A dirt bike racer and a NASCAR driver are at a circular race track. The dirt bike racer challenges the race car driver to see who can get to the opposite end of the track the fastest. The dirt bike will head straight across the diameter and the race car is constrained to the half circle. The top speed of the dirt bike is 90 mph and the top speed of the car is 180 mph. Both vehicles start from a stop at the same point and finish at the same time. The diameter is 1 mile and the race last 60 seconds.[†]



a.) What is the average acceleration of the dirt bike before it reaches its top speed? (Hint: Draw the velocity profile)

$\boxed{3.3\,\text{ft/s}^2}$

b.) How long is the dirt bike accelerating?

$\boxed{40\,\text{sec}}$

c.) How long is the car accelerating?

$\boxed{57.1\,\text{sec}}$

d.) What is the average acceleration of the car before it reaches its top speed?

$\boxed{4.61\,\text{ft/s}^2}$

[†]For this exercise we will assume a constant acceleration. In reality, vehicles have their highest acceleration at low speeds and taper off when reaching maximum speeds.

9 Friction and Acceleration Factors

9.1 Coefficient of Friction

The coefficient of friction, μ, is a ratio of two forces: the tangential force and the normal force. This ratio was presented in Section 10.5 and in Eq. (10.6), and is repeated here:

$$\mu = \frac{F_t}{F_n}.$$

The sliding force acts tangent to the surface on which the object is sliding and is denoted as F_t. This is the force necessary to cause two objects in contact to move with respect to each other. The force holding the objects together acts normal (perpendicular) to the surface and is denoted as F_n. In order to generally find a coefficient of friction between two objects, we will arrange the objects so they are level. Then, the force holding the two objects together is simply the weight of the top object. No other forces are present, especially those that may exist because of a slope.

In traffic crash investigation, we usually need to find the coefficient of friction between the tires of a vehicle and the road surface when the vehicle is braking to maximum.

We should be aware that the coefficient of friction in such cases depends on several things:

1. The composition of the road surface and the tire material that are in contact. Harder tire compounds will generally have lower μ values than will softer tires. In the same way, differing road surfaces will have different values of μ for any given set of tires. Tire construction (radial or bias-ply) may also have an effect.
2. The amount of slip between the tires and the surface. Anti-Lock Braking Systems (ABS) are designed to keep the tire rolling while still providing maximum braking. Tires brake at a maximum when the tire tread velocity is about 80% of the vehicle's translational velocity [4]
3. Any surface contaminants have a potentially large effect on the coefficient of friction. Snow, rain, oil, loose sand, loose gravel, etc, may all effect μ if present in sufficient quantity.

4. The temperature of the tire tread and the road have small effects on μ, usually to the point at which these effects due to temperature may be ignored. In the same way, the sliding velocity between tires and the road have small effects.

5. Tire loading and the area of the contact patch have little effect on μ.

6. By definition, the road slope has *no* effect on the friction coefficient.

9.2 Acceleration (Drag) Factor

An acceleration factor, f, is defined as a ratio of the sum of all forces tangent or parallel to the road surface divided by the weight of the vehicle.

$$f = \frac{\sum F_t}{w} \tag{9.1}$$

We often call this the *drag factor* when discussing slowing vehicles. However, we will see that the acceleration (drag) factor is actually a ratio of accelerations—that of some acceleration a to the acceleration due to gravity g:

$$f = \frac{a}{g} \tag{9.2}$$

We can prove Eq. (9.2) by using Newton's Second Law:

$$F = ma$$

where F is force in pounds or newtons,
m is mass in slugs or kilograms, and
a is the acceleration rate in feet per second^2 or meters per second^2.

Recall from Section 10.3 that Newton's Second Law also defines weight. Weight is a measure of how strongly the earth attracts a given mass. We also know that any object is accelerated toward the earth with a specific acceleration, that due to gravity. We call this acceleration g. So:

$$w = mg$$

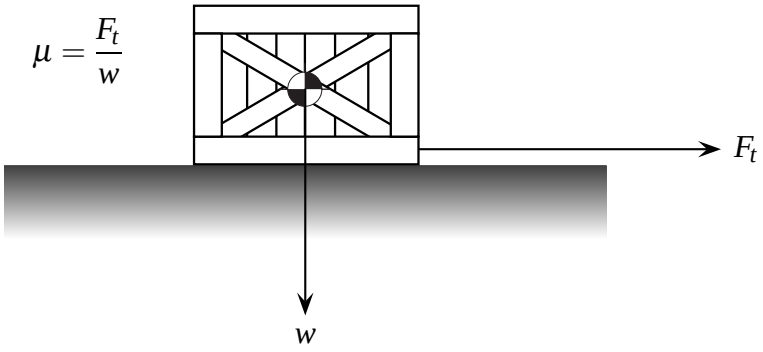


Figure 9.1: An object is pulled across a level surface such that the leading edge does not gouge the surface and the normal force is equal to the weight of the object, $F_n = w$.

or

$$m = \frac{w}{g}$$

Substituting for mass into the equation for Newton's Second Law yields:

$$F = \frac{w}{g}a$$

We may define the acceleration factor as a ratio of force to weight:

$$f = \frac{F}{w} = \frac{1}{w}F$$

where F is the sum of all forces acting parallel to the road. We may substitute the expression for force into this equation as follows:

$$f = \frac{1}{w}\left(\frac{w}{g}a\right)$$

Weight cancels, leaving

$$f = \frac{a}{g}$$

Thus, the drag factor, in addition to being a ratio of forces, is also a ratio of an unknown acceleration to the acceleration of gravity. This concept will help us to describe the motion of the vehicle.

9.3 Determining Drag Factors

In this section, we will assume that the coefficient of friction is known for our system. We need to talk about the coefficient of friction and its relationship to the drag (acceleration) factor. From the following discussion, we will see that the coefficient of friction and the drag factor are similar in definition, but are seldom identical. We will see that the drag factor is a function of the friction coefficient, but may include other influences as well.

We use a lower case f to define the drag factor and the Greek letter μ (mu) to describe the coefficient of friction. The definition of coefficient of friction in words is:

$$\mu = \frac{\text{horizontal force}}{\text{weight}} \tag{9.3}$$

The horizontal force is the force required to move the object over a level surface. The surface must be level so that there is no gravitational component to the pulling force. All parts of the object that are in contact with the surface must be resisting the motion. This is especially applicable to a vehicle, all of whose wheels must be braking to maximum so that every tire is contributing its maximum frictional force. For our purposes, we will consider the friction coefficient to be constant for any two given surfaces and braking conditions (ABS or non-ABS).

The drag factor should be thought of in a slightly different way than the coefficient of friction:

$$f = \frac{\text{All forces parallel to the road surface}}{\text{weight}} \tag{9.4}$$

Like the horizontal force in the definition of coefficient of friction, the force here is parallel to the surface, but the surface does not have to be level and, accordingly, the force does not have to be horizontal. However, if our road is level and all of the wheels brake to maximum, then we can see that the coefficient of friction and the drag factor are identical. Thus, for a level road and all wheels braking:

$$f = \mu \tag{9.5}$$

However, many times our road is not level and/or all of the wheels do not brake. In this case, we cannot use this identity to relate the drag factor with the coefficient of friction. Additional factors need to be taken into account.

Figure 9.2 shows a vehicle stopping on an uphill grade. Therefore, its velocity vector is initially

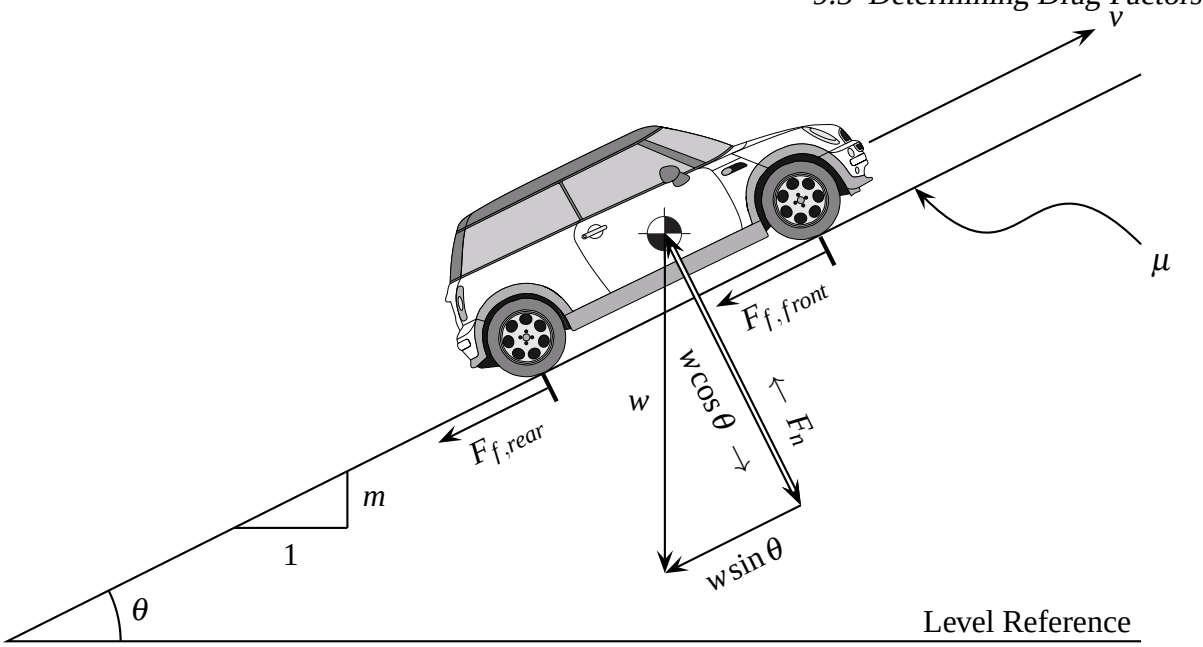


Figure 9.2: A vehicle stopping on an uphill grade. The slope of the grade can be measured as a decimal, $m = \frac{rise}{run}$, or an angle θ. The forces acting on the car include gravity and friction.

heading up hill while the acceleration vector is in the opposite direction. We consider any force acting in the same direction as the acceleration to be positive for this analysis.

Let's look at the forces acting parallel to the road. First, we have the gravitational component, $w\sin\theta$, which is trying to pull the vehicle back down the road. Next, we have the forces from friction that act at the road surface of each tire. These forces are denoted as $F_{f,front}$ and $F_{f,rear}$. The total of all of the friction force is:

$$F_f = F_{f,front} + F_{f,rear} \tag{9.6}$$

If all wheels are braking evenly, then the frictional force can be computed using the definition of the coefficient of friction. Hence,

$$F_f = \mu F_n \tag{9.7}$$

From the geometry shown in Figure 9.2, the normal force from the road surface is expressed as

$$F_n = w\cos\theta \tag{9.8}$$

which gives us an expression for the force due to friction:

$$F_f = \mu w \cos\theta \tag{9.9}$$

However, the sum of all the forces tangent to the road include the component of gravity as well as friction. Thus:

$$F_t = \mu w \cos\theta + w \sin\theta \tag{9.10}$$

We factor out the weight to get:

$$F_t = w(\mu \cos\theta + \sin\theta) \tag{9.11}$$

In Section 9.2 we defined an acceleration (drag) factor as $f = \frac{F_t}{w}$. Making this substitution into Eq. (9.11) gives:

$$f = \frac{\cancel{w}(\mu \cos\theta + \sin\theta)}{\cancel{w}} \tag{9.12}$$

Performing the division gives:

$$\boxed{f = \mu \cos\theta + \sin\theta} \tag{9.13}$$

where f is known as the acceleration or drag factor. This derivation assumes a vehicle was slowing down. If the vehicle was speeding up, then the sign would change on $\sin\theta$. Gravity will always act down hill, so f will be greater if stopping uphill or accelerating down hill.

We can write Eq. (9.13) in terms of the slope, m, instead of the angle, θ, by employing the definitions of cosine and sine.

Consider the triangle that is used to show the slope in Figure 9.2. That triangle has a side adjacent to the angle of length 1 and the side opposite is of length m. Since it is a right triangle, the hypotenuse is determined using the Pythagorean Theorem:

$$hypotenuse = \sqrt{1 + m^2} \tag{9.14}$$

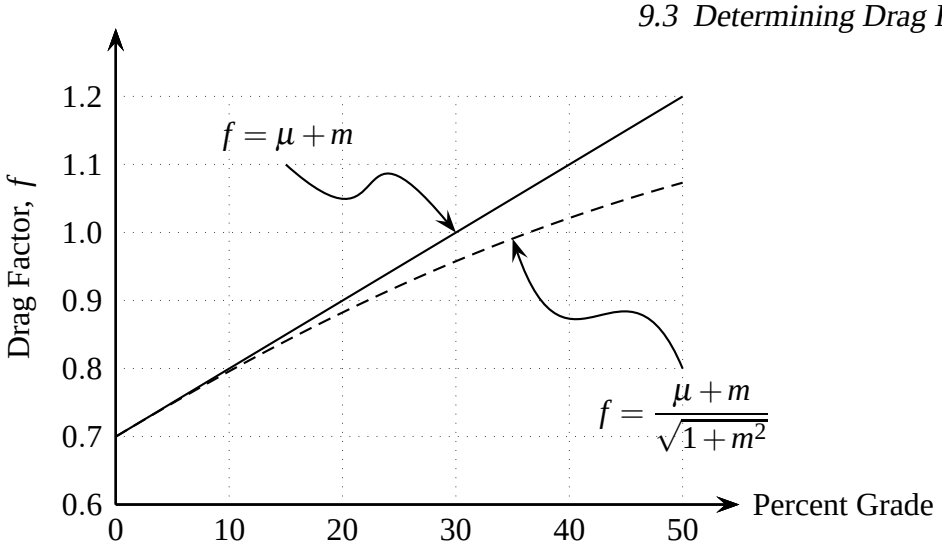


Figure 9.3: A graph showing the drag factor adjusted for the slope of the surface with a coefficient of friction of 0.70. Equation (9.13) or (9.17) is the lower dashed line and the graph of Eq. (9.18) is the upper solid line. The graph of the two equations separate once $m > 10\%$, which indicates that the simplification of Eq. (9.18) no longer applies.

The cosine of an angle is the side adjacent over the hypotenuse,

$$\cos\theta = \frac{1}{\sqrt{1+m^2}} \tag{9.15}$$

and the sine of an angle is the side opposite over the hypotenuse,

$$\sin\theta = \frac{m}{\sqrt{1+m^2}} \tag{9.16}$$

Substituting Eqs. (9.15-9.16) into Eq. (9.13) gives an alternative way of writing the drag factor as a function of slope:

$$\boxed{f = \frac{\mu+m}{\sqrt{1+m^2}}} \tag{9.17}$$

For small slopes, $\sqrt{1+m^2} \approx 1$, so we can write

$$\boxed{f = \mu + m} \tag{9.18}$$

This is a valid assumption as long as the slope is less than 10%. This limitation is illustrated with the graph in Figure 9.3.

9.4 Effects of Uneven Braking on Drag Factor

9.4.1 Definition of Percentage of Braking

There are times when all of the weight-bearing wheels of a vehicle are not braking because of vehicle damage or poor maintenance. In such situations, we may define a *percentage of braking*, n, which is simply the percentage of the vehicle weight borne by the braking wheels compared to the total weight of the vehicle. This is written mathematically as:

$$n = \frac{\text{weight on braked wheels}}{\text{total weight}} \tag{9.19}$$

Example 9.1. Determine the value of n for a 6000-lb pickup pulling a 2000-lb trailer when the trailer has no brakes.

We will assume all the brakes on the truck are working. However, the load on the truck's tires is a combination of the truck weight and the tongue weight of the trailer. The tongue weight of the trailer as configured is 400 lbs. Therefore, the weight on the braked wheels is 6400 lb and the total weight is 8000 lb. This gives

$$n = \frac{6400}{8000} = 0.80$$

It is noteworthy that any weight shift during braking would result in more weight on the braked wheels and increase n. Therefore, neglecting weight shift gives a lower value of n, which, in turn, will give a lower speed in a slide-to-stop situation.

Theoretically, the concept of percentage of braking can be applied to an individual tire. This reduction in the braking force for an individual tire is sometimes called a "lock-up" factor. At issue with this logic is that evidence is rarely left by tires that are not performing near their maximum capability and any sort of "lock-up" factor is subjective and may be difficult to determine at the time of a crash. We will explore the "lock-up" factor more completely in Chapter **??**.

9.4.2 Application of Percentage of Braking

The percentage of braking, n, is multiplied by the friction coefficient to assist in determining the proper value for the acceleration (drag) factor in the following way:

$$\boxed{f = \frac{\mu n + m}{\sqrt{1+m^2}}} \tag{9.20}$$

Whenever we perform a drag factor calculation, we calculate it independently of the equation in which it is used. For example, if we are performing a speed calculation that relies upon the drag factor, we will do two sets of calculations. The first is a drag factor calculation and the second is the speed calculation. When we look at a problem in the suggested fashion, we increase our understanding of the relationship between the drag factor and the coefficient of friction. The following equations illustrate this relationship for several different types of cases; the last three also include braking percentages n:*

1. Drag factor equation for uphill grades of 10% or less, all wheels braking. Use minus m if downhill.

$$f = \mu + m \tag{9.21}$$

2. Drag factor equation for uphill grades greater than 10%, all wheels braking. Use minus m if downhill.

$$f = \frac{\mu + m}{\sqrt{1+m^2}} \tag{9.22}$$

3. Drag factor accounting for grade and percentage of braking. Use minus m if downhill.

$$f = \mu n + m \tag{9.23}$$

4. Drag factor for two side-by-side level surfaces with braking percentage.

$$f = \mu_1 n_1 + \mu_2 n_2 \tag{9.24}$$

*These equations are representative of some common situations. The general drag factor equation may be modified to account for other situations as defined by the evidence gathered at the scene of the crash.

5. Drag factor for two side-by-side surfaces with grade and braking percentage. Use minus m if downhill.

$$f = (\mu_1 n_1 + \mu_2 n_2) + m \tag{9.25}$$

Our independent calculations of the drag factor f would entail entering whatever data that might be required into the equation appropriate to the particular case.

9.5 Gathering Road Friction Data

When investigating crashes, one of the essential elements needed is the friction of the roadway. Obtaining an accurate representation of this friction value is important. Various methodologies are used by investigators to obtain roadway friction. Depending on the chosen methodology, an accurate friction value may or may not result. The various methodologies are discussed below in order of increasing accuracy and precision.

9.5.1 Tables

One may tend to rely upon published tables that represent work of other investigators at other times and places; for examples, see Refs. [5–7]. Tables will generally have a rather large range of values for any given surface and condition. Since we often do not know the circumstances of the testing procedure, we do not know specifically how a particular test corresponds to the circumstances of our crash. Tables and charts are more useful in unusual situations in which we may have a difficult time conducting tests, such as the drag factor for large animals, a vehicle sliding on its top or sides, snow penetration, etc. Tables can also provide a “sanity check” for your own testing.

9.5.2 Drag Sleds

A drag sled is a device used to determine the friction between tire rubber and a pavement surface. The drag sleds generally come in two configurations. In one, a box is filled with concrete or other heavy material, an eye bolt is horizontally attached to an end of the box at the approximate height of the center of mass, and a section of tire tread is affixed to the bottom of the box. A sled of this type is shown in Figure 9.4. These sleds are called *flat bottom* sleds.

(a) Drag sled being weighed with the same scale as will be used to pull it.

(b) Drag sled being pulled across the pavement surface.

Figure 9.4: The weight of a drag sled must be known before testing a surface for the drag factor. Drag sled operators must be familiar with their equipment to get consistent and accurate measurements. (Russell Strickland Photos)

Another common drag sled is made from a section of a whole tire that has been cut out in a semi-circle. Bolts are placed through the tire sidewall to help the tire section keep its shape, and an eye bolt is horizontally inserted through the tread section at the approximate height of the center of mass of the tire section. The tire section is then filled with concrete. This is the *round bottom* configuration.

In use, the sled is weighed with a spring scale, as in Figure 9.4. The same scale is used to pull the sled across the surface, and the pull force is noted. The drag factor for the surface, including contributions due to slope, is calculated with the following equation:

$$f = \frac{F}{w} \tag{9.26}$$

Flat bottom sleds must be pulled at a consistent or prescribed angle (unique to that sled) to the surface, whereas a round bottom sled is pulled parallel to the surface.

Drag sleds have an advantage in that they are quick and easy to use. An experienced operator may develop a technique so that respective pulls on the same surface are precise, meaning the same or similar results are obtained regularly. Unfortunately, this precision does not usually extend to two or more operators pulling their respective sleds over the same surface. Each operator may obtain

consistent results, but each of their results may be quite different, even on the same surface. Hence, the accuracy of the method is called into question.

Drag sleds measure the friction that exists between a non-deformable tire tread and the pavement. There are no brake system effects measured, such as latency or balance. Drag sled usefulness is compromised on anything except dry, hard pavement surfaces. For example, a drag sled pulled across the top of damp grass on a road edge may give an artificially low drag factor if the crash vehicle furrowed the surface. A drag sled always has a braking percentage, *n*, of 100%. A drag sled provides primarily a sliding drag factor, so its use for testing a surface upon which an ABS equipped vehicle has skidded is questionable. Some investigators will attempt to model ABS braking behavior by taking the pulling force required to break loose the sled (i.e., just enough force to get it to move) and using that force in Eq. (9.26). This method does not model the ABS algorithm used by the vehicle, and may or may not be an accurate representation of actual vehicle behavior. We have no way of knowing this if the drag sled is the only means used to gather friction data.

9.5.3 Tests with Skidding Vehicles

We find it useful in crash reconstruction to do vehicle skid testing whenever possible. Ideally, we would like to be able to test the actual vehicle involved in the crash. In the case of bicycle/pedestrian collisions, this may be possible. As damage to the vehicle increases in severity, this may not be possible. In such cases, it is advisable to locate an exemplar vehicle for testing. Keep in mind that we are testing a system, rather than just finding the amount of friction between rubber and a pavement surface. Brake system latency, brake system balance, and tire effects are all included with vehicle skid testing.

9.5.3.1 Measured Test Skids

Test skidding vehicles to determine road friction characteristics is a method that has long been used in crash reconstruction. The vehicle is brought up to a known speed, usually measured with radar, and the brakes are applied as hard and fast as possible. This should cause the wheels to lock on a non-ABS vehicle with a maintained brake system. Characteristic locked-wheel skid marks will be produced. The dark portion of the skid mark is where the wheel has locked and is sliding. The lighter portion of the skid mark, known as the shadow mark, is the portion of the skid mark made just before the wheel locks. This is the portion of the skid mark that contains maximum braking. The shadow is included when measuring the length of the skid mark.

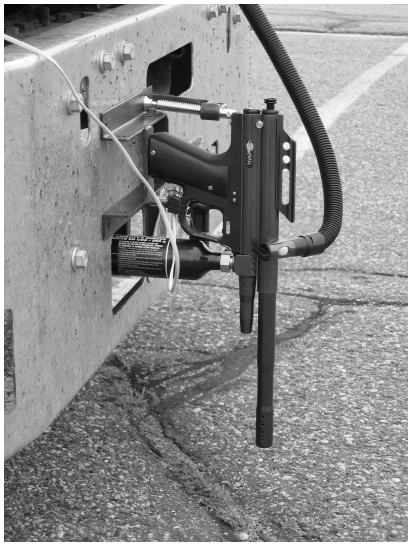
The investigator should leave the vehicle stopped on its skid marks or mark the tire location on the pavement. The investigator must then locate the beginning of the shadow portion of the skid. If the vehicle stops straight, it may not be possible to differentiate between the front and rear skid marks since they will overlap. In such cases, the investigator would identify the beginning of both the right and left skid marks and would then measure to the corresponding rear wheels to determine the respective skid length. If the investigator is testing a pickup truck with rear ABS only, the investigator would identify and measure the front wheel skid lengths. These measurements are then averaged to determine a skid length for use in a computation.

The test drag factor may be computed with the following equation:

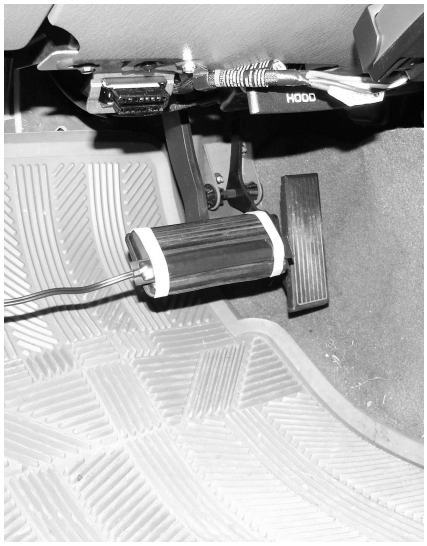
US		SI	
$f = \frac{S^2}{30d}$	(9.27)	$f = \frac{S^2}{254d}$	(9.28)

where d is distance in feet (meters),
f is the acceleration factor, and
S is the speed in miles per hour (kilometers per hour).

There are some disadvantages with this method. Besides the obvious, such as being on the road and the safety considerations, we may see that the result is sensitive to the speed. Because the speed is squared in the equation, a small error in speed will propagate into a larger error when determining the drag factor. Another possible source of error is when determining the skid length by the inspection of the shadow portion of the skid. Pavement marking may vary in prominence depending on the surface and it may be difficult for the investigator to locate the actual beginning of the shadow mark. If the test skid is measured too short, then the drag factor computation will be too high. Additionally, this traditional method has no way of determining brake system latency. This method is usable for ABS-equipped vehicles if the deceleration scuff marks (ABS skids) are discernible enough to measure.



(a) Shot marker attached to the front of a bumper.



(b) Tape switch on a brake pedal.

Figure 9.5: Shown here is a shot marker system built by the authors that will fire a paintball into the pavement as soon as the driver touches the brake pedal. (Jeremy Daily Photos)

9.5.3.2 Test Skids with Shot Markers (Bumper Guns)

A *shot marker*, also known as a *bumper gun*, is a device attached firmly to the bumper of a test vehicle that will fire a marker pellet onto the pavement surface as soon as the brake pedal is touched by the driver's foot. This is accomplished by means of a tape switch attached to the brake pedal that activates the gun. The time for a paintball-based marker to impact the pavement from the moment the brake pedal is touched typically runs between 30 ms to 50 ms.

To use, the system is set up on the test vehicle, which is then accelerated to a known speed. The brake is applied hard and fast, firing the marker onto the pavement. The vehicle is brought to a stop and the distance from the pavement mark to the barrel of the shot marker is measured. The drag factor is then calculated with Eq. (9.27) or (9.28).

The shot marker system enables the investigator to measure from an easily identifiable point (the pavement mark) rather than from a less defined skid mark beginning. Hence, any error associated with determining the beginning of a skid mark is essentially eliminated. However, the method still requires the speed to be known accurately, and the resultant calculation is subject to the same errors as is the skid mark measuring method. Also, the drag factor resulting from the shot marker distance will be conservative in that the shot mark will appear on the pavement before visible tire marks.

Because the shot mark appears on the pavement before any measurable skid marks, it is possible to get an idea of the brake system latency for the vehicle. The distance from the shot mark to the beginning of the skid mark is measured. The vehicle will be slowing very little over this distance, so a constant velocity for that distance may be assumed. We may calculate the brake latency (lag time) with the following equation:

US	SI
$t = \frac{d}{1.466S}$ (9.29)	$t = \frac{3.6d}{S}$ (9.30)

where t is time in seconds,
d is distance in feet (meters), and
S is speed in miles per hour (kilometers per hour).

We could add the 30 ms to 50 ms time for the marker to hit the pavement from the moment of firing, thus getting an idea of how much time elapsed between the touching of the brake pedal and the appearance of measurable skid marks.

9.5.3.3 Test Skids with Recording Radar

The test vehicle may be set up with a recording radar unit such as the Stalker® radar system. This setup may be either interior or exterior to the vehicle. The radar instrument then gathers continuous speed data for the test, whether the vehicle is accelerating, driving at constant speed, or decelerating. Such instrumentation is useful for skid tests, acceleration tests, and coast down tests. The radar unit measures speed, and then acceleration may be calculated after downloading the information into a computer and applying the appropriate math. This technique does not involve measuring skid

Figure 9.6: Stalker® radar used to record the velocity history of a vehicle. (Wade Bartlett Photo)

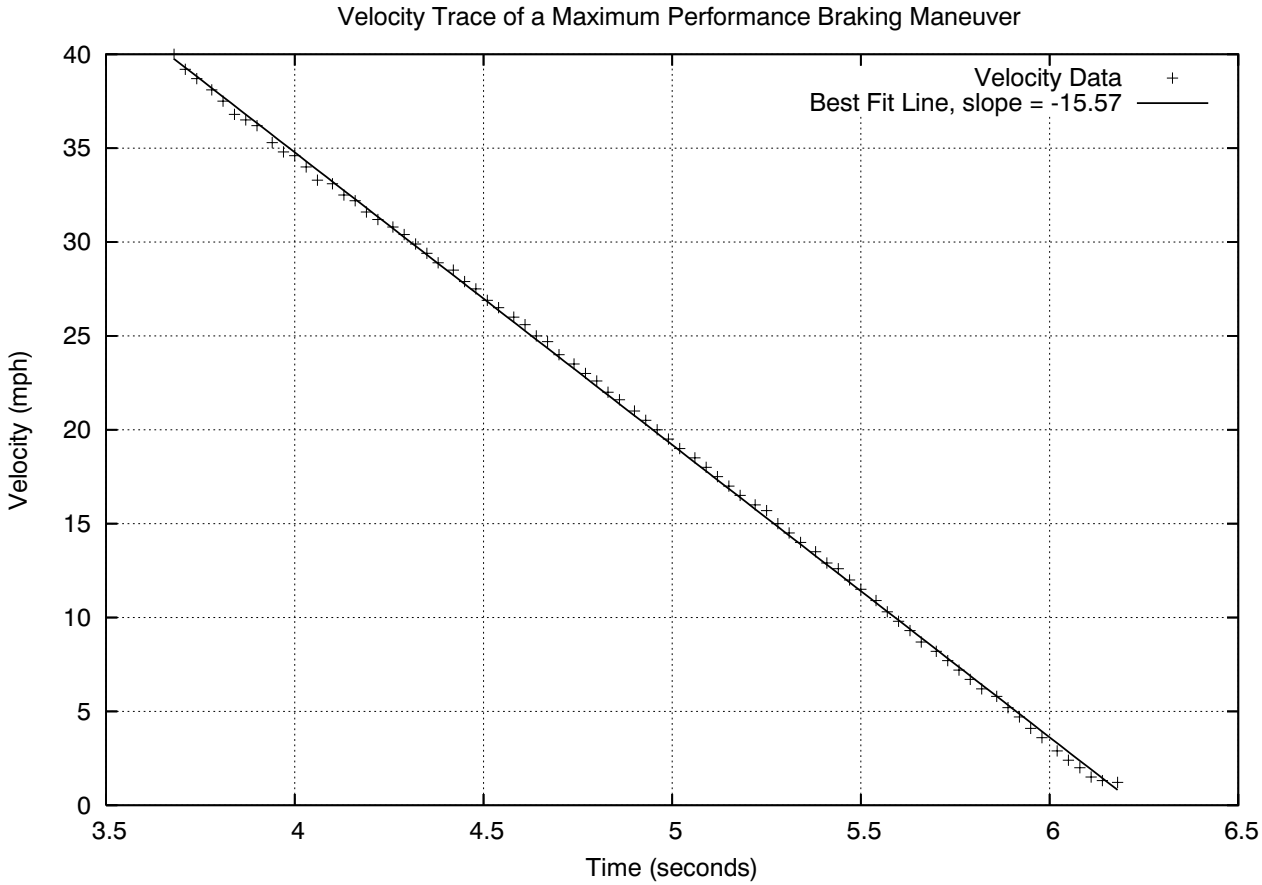


Figure 9.7: An example of a velocity trace acquired from a Stalker® radar during a skid test. The drag factor is computed by converting the slope from units of mph/sec to ft/sec^2 and dividing by $g = 32.2$ ft/sec^2. In this example, -15.57 mph/sec equals -22.83 ft/sec/sec, which gives a value for f of -0.708. A similar unit conversion would be performed for the SI system. (Data courtesy of Wade Bartlett.)

marks, so the uncertainty in determining the beginning of the skid mark is eliminated. An example of the results from the data obtained from a Stalker® radar is shown in Figure 9.7.

9.5.3.4 Test Skids with Accelerometers

Accelerometers are measuring instruments that may be placed in or on vehicles to accurately and precisely measure the acceleration of the vehicle at any moment in time. This accelera-

tion may be positive or negative. Negative acceleration is termed deceleration, and is what we desire to know with a stopping or skidding vehicle. Some accelerometers, such as the Vericom VC3000DAQ™, also measure lateral acceleration and calculate both speed and distance from the acceleration measured.

Accelerometer output includes skid distance, skid time, speed at the beginning of the skid, and the average drag factor for the test. The acceleration is measured, from which speed and distance are computed. In addition, software available with the accelerometer enables the investigator to examine the shape of the acceleration curve (Figure 9.9) and toggle between any two points on the graph, enabling the computation of a drag factor over selected portions of the skid.

Accelerometers are both accurate and precise. The results obtained are repeatable, often to the nearest hundredth of a g. The results are accurate *for the vehicle being tested.* This is an important point, and illustrates why we wish to do our skid testing either with the vehicle involved in the crash or an exemplar vehicle. For example, let us say the investigator's vehicle is a late model police car and the crash vehicle is a late model heavy-duty pickup truck. Even with all other factors being equal, such as properly adjusted and working brake systems, it would not be uncommon for the tested drag factor for the car to be 0.10 to 0.15 higher than that measured for the pickup. If possible, perform several accelerometer tests with the crash vehicle or an exemplar vehicle.

Performing test skids with an accelerometer does not involve measuring skid marks, so the uncertainty in determining the beginning of the skid mark is eliminated.

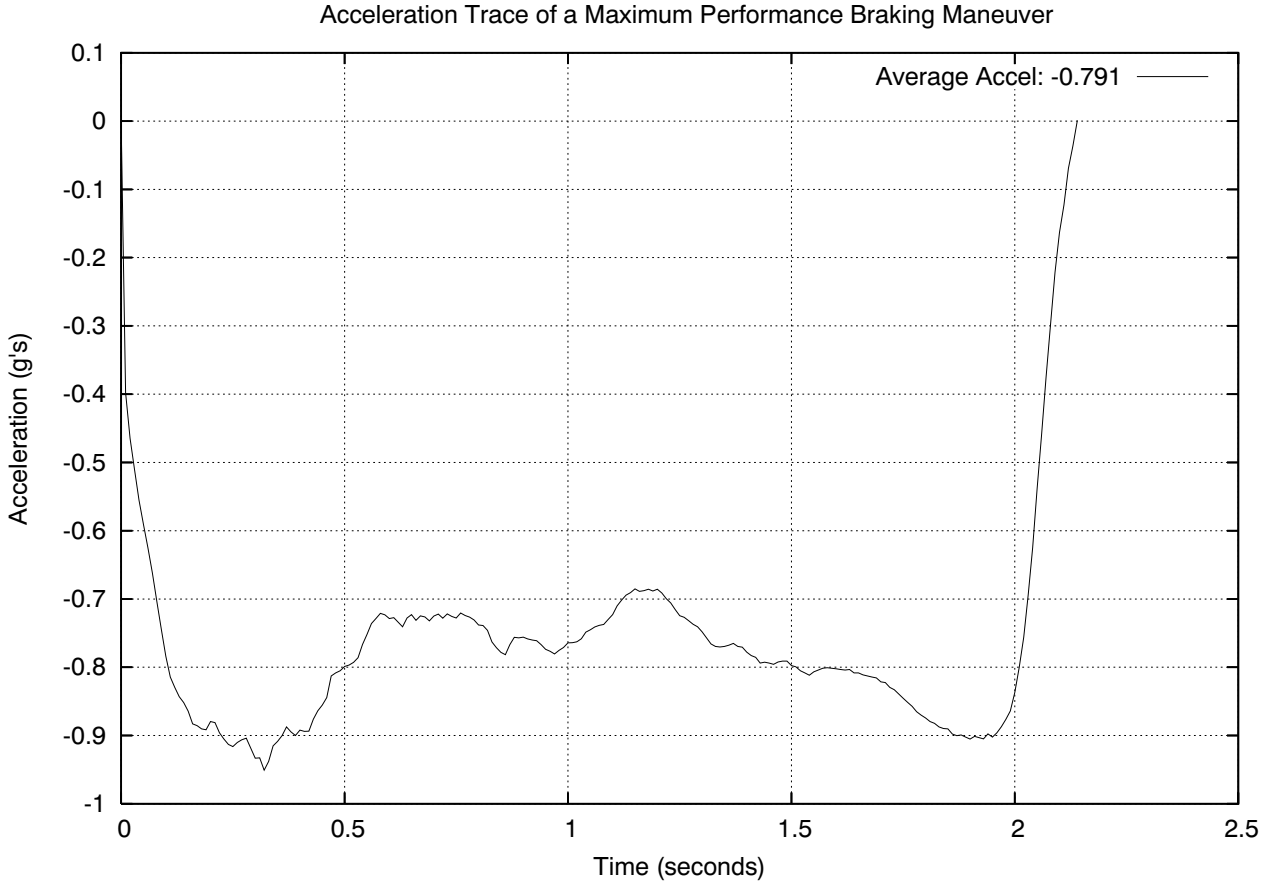


Figure 9.9: This graph shows an example of the data obtained from an instrumented skid test. This particular graph was made from data obtained with a Vericom VC3000DAQ™ braking computer mounted to the windshield on May 19, 2005. The vehicle was a 2004 Chevrolet Impala without anti-lock brakes. The surface was an asphalt parking lot in Arlington Heights, Illinois.

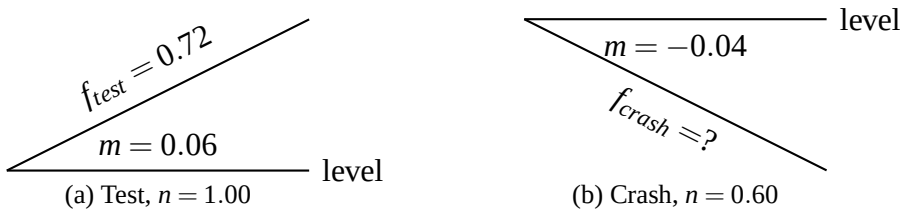
9.6 Determining Drag Factors from Test Data

When we are reconstructing a traffic crash, we need to determine the drag factor(s) of the surface over which the vehicle traveled. We will look at how to analyze data from test skids.

Let's say we have a situation in which we cannot do a skid test at the actual crash site. We choose a similar surface composition, tires, and vehicle for our test, but we find that the slope is different.

Example 9.2. What is the drag factor for a crash that occurred on a 4% downslope with only 60% braking efficiency?

The first step is to perform an actual skid test. The test results give $f_{test} = 0.72$ on an uphill slope of 6%. All brakes worked for the skid test, so the braking efficiency is 100%. The situation at hand is illustrated in Figure 9.10.



(a) Test, $n = 1.00$ (b) Crash, $n = 0.60$

Figure 9.10: The two different drag factors can be related based on the different slopes.

First, use the test information to find the coefficient of friction μ:

$$f_{test} = \mu n + m$$
$$0.72 = \mu(1) + 0.06$$

Solve for μ:

$$\mu = 0.72 - 0.06$$
$$= 0.66$$

Now solve for f_{crash}:

$$\begin{aligned} f_{crash} &= \mu n + m \\ &= 0.66(0.60) + (-0.04) \\ &= 0.35 \end{aligned}$$

This method should be used regardless of the type of test used to determine the test drag factors. Shortcut methods may inadvertently multiply the slope of the road by a percentage of braking, which is physically incorrect because it will result in taking a percentage of gravity. Since gravity is fixed, mother nature will not let us change its value!

With acceleration testing, we are testing a system that consists of the tires, vehicle, and surface. If the vehicle is equipped with anti-lock brakes, then we are testing the braking system as well.

9.7 The Friction Circle and Lateral Friction

A tire can generate frictional forces in every direction. So far, we have only discussed friction in the longitudinal direction or along the vehicle's *x*-axis. For a vehicle to turn, the tires must also generate friction to induce the lateral acceleration. In the cases of both turning and braking, a tire can generate frictional forces in both the *x*- and *y*-directions. However, the performance of the tire in one direction is degraded by the demands for performance in the other. The relationship of frictional forces in the two directions follows the concept of a circle.

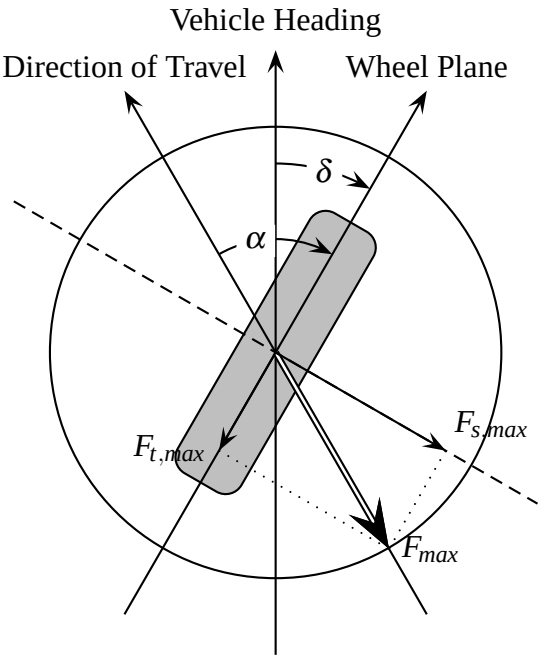


Figure 9.11: The friction circle showing a locked sliding wheel.

The concept of the friction circle demonstrates that the maximum frictional force developed by a tire has a constant magnitude, regardless of direction. This means the maximum friction can be described by a circle that has a radius equal to the maximum frictional force. Therefore, if we know the frictional force in one direction,

then the magnitude of the force in the other direction can be determined using the properties of vectors.

A tire that is locked and skidding generates a frictional force that is opposite the relative velocity of the skidding tire and the road. Also, a locked tire does not generate any cornering or lateral force. This explains why skidding vehicles cannot steer when their front wheels are locked. Furthermore, if all tires are locked, the vehicle will tend to track down the *super-elevation*[†] when skidding.

Finally, the lack of lateral stability from a locked wheel explains the phenomena of the "e-brake" turn when a driver locks the rear wheels and is able to skid the back of the vehicle around. This can also occur when the tires break free from the surface because too much torque is applied to the rear axles. This instability can be remembered with a simple phrase: "The locked wheels will lead."

[†]Super-elevation is the cross-slope of the road or the bank in a curve.

Exercises

Exercise 9.1. A crash vehicle is skidding downhill on a 3% grade. Its weight distribution is 57% front and 43% rear. One front wheel is not braking. A test skid with a similar vehicle with all brakes working is conducted on a similar surface on a downhill grade of 8%. The drag factor for this test is calculated to be 0.64. What is the drag factor of the crash vehicle?

$\boxed{f = 0.48}$

Exercise 9.2. A crash vehicle skids on a level surface. Its percentage of braking is 68%. An exemplar vehicle is used for the test skids, yielding a testing drag factor of 0.61. This exemplar vehicle has a braking percentage of 82%. The test surface was level. What is the drag factor for the crash vehicle?

$\boxed{f = 0.50}$

Exercise 9.3. A crash vehicle skids with a braking percentage of 50% up a 6% slope. Your test skid with a similar vehicle and 100% braking on a slope of -4% yields a drag factor of 0.69. What is the drag factor of the crash vehicle?

$\boxed{f = 0.42}$

Exercise 9.4. A crash vehicle skids downhill on a -8% grade. Its percentage of braking is 60%. Your test skid involves the use of a similar vehicle in the same direction as the crash vehicle. The test vehicle has a braking percentage of 100% and yields a test f of 0.67. What is the drag factor of the crash vehicle?

$\boxed{f = 0.37}$

Exercise 9.5. Consider the same information in Exercise 9.4. Now our test vehicle has a percentage of braking of 80%, yielding the same 0.67 test value for f. What is the drag factor of the crash vehicle?

$\boxed{f = 0.47}$

Exercise 9.6. A crash vehicle skids up a 12° slope. Its percentage of braking is 72%. A similar test vehicle with 100% braking has skidded down a 4% slope, yielding a test drag factor of 0.41. What is the drag factor of the crash vehicle?

$\boxed{f = 0.52}$

10 Dynamics and Newton's Laws of Motion

Dynamics is the study of motion with regard to what caused the motion; it is the study of the relationship between motion and the influences that cause the motion. When we look at traffic crashes, we are primarily concerned with the dynamics of motion. This is in contrast to kinematics, which only describes the motion without regarding its cause.

Many years ago there lived an eccentric genius by the name of Sir Isaac Newton. According to the legend, Newton was sitting under an apple tree one day when an apple fell on his head. This incident supposedly provided the inspiration for his laws of motion and gravitation. In reality, the process took a little longer. Nevertheless, Newton is generally recognized as the father of mechanics. We will be studying Newtonian mechanics, which is the study of motion of discrete masses.

10.1 Newton's First Law

Newton's First Law may be stated like this:

> Every body at rest tends to remain at rest, whereas every body in motion tends to remain in motion, unless it is acted upon by an unbalanced external force.

This law describes the concept of inertia. We are all familiar with inertia. If we are riding in a car that has to stop suddenly, it is the inertia of our bodies that tends to resist that stopping, and we feel ourselves slide forward against our seat belts. (We do wear seat belts, of course?)

We will see how this concept of inertia may help us to understand momentum and the forces that result when the momentum of a body changes.

10.2 Newton's Second Law

Newton's Second Law may be stated like this:

> The acceleration of any body is directly proportional to the force acting on the body, whereas it is inversely proportional to the mass of the body.

Before we discuss the application of this law, we need to define some terms. To begin with, what is mass? Mass is different from weight. A few simple definitions:

Mass is the amount of matter that an object contains.

Weight on the other hand, is the measure of how strongly the earth attracts an object of a given mass.

If we were to take an object of a given mass, say a block of steel, up to the moon, it would weigh less than it does down here on earth because the gravitational attraction would be less. However, the mass of the block would be the same on the moon as on the earth because the block of steel would still contain the same amount of matter. This distinction will become important to us in later discussions.

Recall the definition of acceleration from Section 7.1 on page 7-2:

Acceleration is the rate of change of velocity over a period of time.

In other words, acceleration is the measurement of how fast the velocity of an object changes. This change can be either positive or negative. If the acceleration is positive, the object will speed up. If the acceleration is negative, the object will slow down. Acceleration will also result if the direction component of the velocity changes.

Now that we have reviewed the ideas of mass, acceleration, and velocity, let's talk about the terms "directly proportional" and "inversely proportional." In a very basic sense, if we have something, call it A, that is directly proportional to something else, call it B, then an increase in B will result in an increase in A. The first part of Newton's Second Law states that the acceleration of a body is directly proportional to the force acting on the body. According to this law, if we increase the force acting on the body, the acceleration of the body will increase. From this same law we can infer that if the acceleration of the body has been increased, then the force acting on that body must also have been increased.

Now on the other hand, if we have something, call it C, that is inversely proportional to something else, call it D, then an increase in D will result in a decrease in C. The second part of Newton's

Second Law states that the acceleration of an object is inversely proportional to the mass of an object. Thus, if we increase the mass of the object, with no changes in the force, the acceleration will decrease.

Everything that has been just discussed is described mathematically in a simple, but very important, equation that expresses Newton's Second Law:

$$a = \frac{F}{m} \tag{10.1}$$

where a is acceleration, F is the external unbalanced force acting on the body, and m is the mass of the body.

We can see that if F gets bigger, then so does a if m remains the same. Similarly, if m gets bigger, then a gets smaller if F remains the same. This equation will be very useful to us later when we derive the equations relating the speed of a vehicle to the length of skid-marks left by the vehicle.

A more conventional form of Eq. (10.1) is:

$$F = ma \tag{10.2}$$

10.3 The Concepts of Mass and Weight

Newton's Second Law will help us to further understand the concept of weight. We said that weight is a measure of how strongly the earth attracts an object of a given mass. Viewed this way, weight is seen to be a force. We may therefore substitute weight into our equation for Newton's Second Law so that it replaces F:

$$w = ma \tag{10.3}$$

The weight of an object of any given mass varies very little from place to place on the surface of the earth. Additionally, we already know that the mass of the object is the same wherever it is, since the same amount of matter is present. Therefore, a in this equation is a very specific acceleration. It is the acceleration caused by the force of the earth's gravity acting on any given mass. This acceleration, which is usually denoted by the letter g, has a value of approximately 32.2 feet per second per second (9.81 meters per second per second). We may then rewrite the above equation

substituting g for a:

$$w = mg \tag{10.4}$$

This is the equation by which we can determine the relationship between mass and weight. We will find the equation more useful in some of our later derivations if it is written in this form:

$$m = \frac{w}{g} \tag{10.5}$$

10.4 Newton's Third Law

Newton's Third Law has to do with the relationship between acting and reacting forces. It may be stated as follows:

> For every force exerted on a body by another body, there is an equal but opposite force reacting on the first body by the second.

Another way to say this: For every action, there is an equal but opposite reaction. Let's explore this further: Let's say there is a five pound weight sitting on a table. The weight pushes down on the table with a force of five pounds. But similarly, the table pushes up on the weight with a force of five pounds. Or, as another example, if we pull or push on a car, the car in turn pushes or pulls back on us with the same force. Finally, a simple dynamic example of this law of motion is provided by a whirly-gig lawn sprinkler. The water leaving the arms of the sprinkler exerts a force on the arms and makes them rotate.

In the field of crash investigation, acting and reacting forces are found when a vehicle skids to a stop. The tires exert a force on the road while the road, in turn, exerts a force on the tires that is equal in magnitude, but opposite in direction. Newton's Third Law also states that the collision forces acting on vehicles are equal in magnitude and opposite in direction.

10.5 The Concept of Friction

No discussion of basic physics would be complete without a section on friction and its effects on an object. Although friction is not one of the laws of motion, we should probably have a look at it before we go any farther.

Friction is one of those things that you either wish you had or did not have. If you are driving your car and have to stop quickly, you want to have as much friction as possible. Conversely, to get maximum mileage from your car, you want the rolling friction of the tires on the road surface and the friction in the engine to be as little as possible.

Just what is friction? One definition is:

Friction is the resisting force that has to be overcome if one body is to move with respect to another body with which it is in contact.

There can be friction due to air resistance, friction due to rolling, friction due to the sliding of one surface over another, or friction between two surfaces just resting together. There are other forms of friction as well, but we will not discuss them. Neither will we discuss any of the theories on why friction exists. Some of those theories get a bit complicated.

We are going to talk about the friction between two surfaces resting together and the friction between two surfaces when one is sliding over the other. The first kind of friction is usually referred to as *static* friction and the second kind of friction is usually referred to as *kinetic* or *dynamic* friction.

There are several observations that have been made over the centuries concerning friction. We will list some of these observations and will then discuss them further. One observation is that the force of friction between two objects is dependent on the materials of the two objects that are sliding (one surface sliding over another surface). The area of the sliding surface is really not that important. Another observation is that the force of friction between two objects depends on the amount of force holding the objects together. It is also important for us to know as crash investigators that the force of friction between two surfaces may be the same for a wide range of velocities. This allows us to conduct a test skid at a lower speed than the suspected speed of the vehicle involved in an accident. There are some small variations to this rule, but this is one of the things we learn when we are taught how to do test skids. One other observation that has been made concerning friction is that the force of the friction of a body sliding over another at a constant speed is less than the force required to get the body moving initially. In other words, dynamic friction is not as great as static friction.

Now that we understand these observations about the nature of friction, we need to consider the *coefficient of friction*. This is simply the ratio of the force that will cause one of the bodies to slide over the other to the force that is holding the bodies together. We can express this ratio in the form of an equation:

$$\mu = \frac{F_t}{F_n} \tag{10.6}$$

where μ is the coefficient of friction, F_t is the force required to slide one of the bodies over the other, and F_n is the force that is holding the bodies together. The subscript $_n$ denotes that the force acts normal to, or perpendicular to, the surface. The Greek letter μ(mu) is used to denote the coefficient of friction. The value of μ is determined experimentally.

10.6 Free Body Diagrams

When analyzing objects, we are very well served to draw a sketch of the object with all the forces acting upon it and the dimensions applicable to it in a defined coordinate system. This is called a free body diagram, which a drawing representing a real system that is used to understand the physics influencing its behavior.

10.6.1 Free Body Diagram Example

Draw a free body diagram of a box sitting on a table, as shown in Figure 10.1 on page 10-7. If we want to draw a free body diagram of the box, we have to consider two forces: the weight of the box and the force of the table. Since the box is not accelerating, the forces must be balanced. This is an application of Newton's Third Law, in which the action is the weight of the box and the reaction is the pressure from the table. Remember, pressure is force divided by area. In Figure 10.1b, there is a coordinate axis showing the positive directions. According to the definition of positive directions, the weight is acting in a negative direction and the pressure is positive. We can also represent the pressure as a force acting at the center of the pressure distribution. This is important because we will later simplify the tire contact patch pressure as a point force acting in the center of the contact patch.

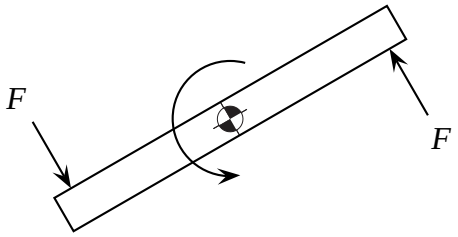
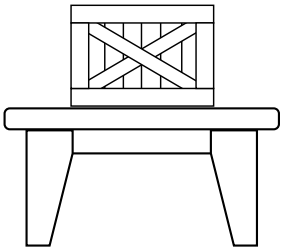
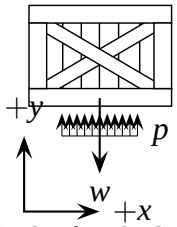


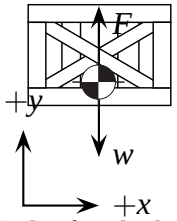
Figure 10.2: A couple is two forces of equal magnitude acting in opposite directions, which act to rotate a body.



(a) A real system of a box on a table.



(b) The free body diagram of the box using pressure.



(c) The free body diagram of the box using forces.

Figure 10.1: The representation of a box resting on a table using a free body diagram. The weight of the box w is equilibrated by the pressure p exerted by the table. The coordinate system in the corner of subfigures (b) and (c) shows positive directions.

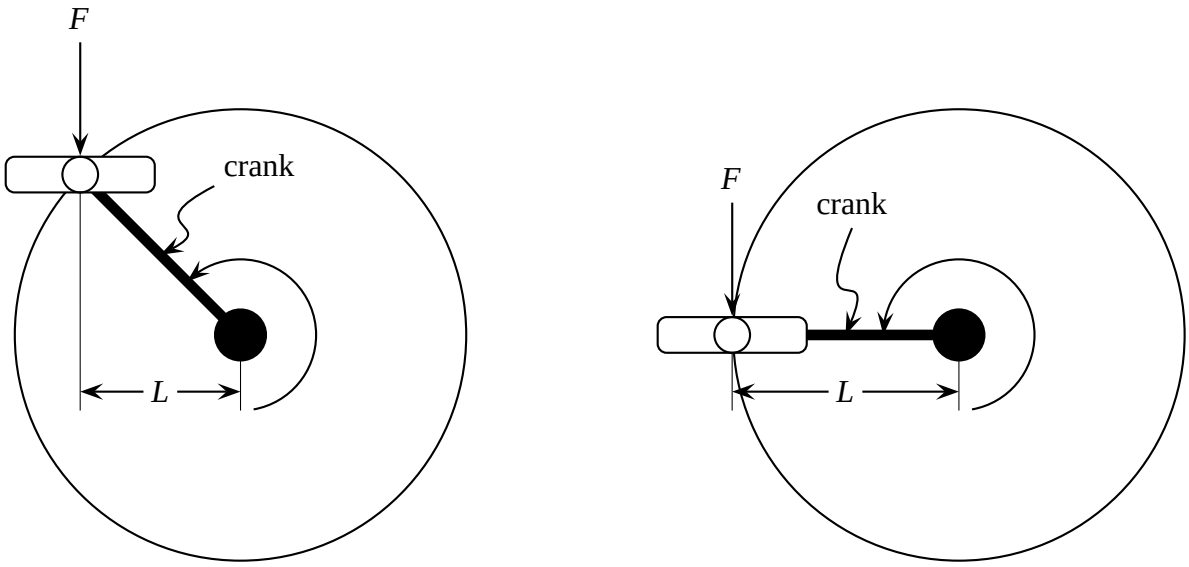
10.7 The Concept of Torque

Let's begin this section by defining torque. *Torque* is a twisting effort. Often we will use the term *moment* because a torque and a moment are the same thing. Torque is produced by a couple, which is simply defined as two forces, equal in magnitude and opposite in direction, that act to rotate a body. The symbol for torque is τ, the Greek letter tau. Other times we may use the symbol M to denote a moment. Both have the same meaning.

> The magnitude of the torque produced is a force multiplied by the length of the lever arm, or moment arm, against which the force is acting.

In mathematical form:

$$\boldsymbol{\tau} = \mathbf{F} \times \mathbf{d} \tag{10.7}$$



(a) The lever arm is less than the crank length.

(b) The lever arm is equal to the crank length.

Figure 10.3: A bicycle crank illustrates the concept of torque. The torque about the center is the force times the perpendicular distance of the force from the center.

where $\boldsymbol{\tau}$ is the torque vector, **F** is the force vector, and **d** is the lever arm vector that starts at the center of rotation and ends where the force is applied. The magnitude of the torque is the magnitude of the force, multiplied by the perpendicular distance the line of action is away from the center. A lever arm or moment arm can be described as a line, or lever, extending out from the center of mass of the object, perpendicular to the force being applied, and against which the force is operating.

For example, if we are riding a bicycle, we have to push on the pedals to make the bicycle move. If we push straight down on the pedal, then we will be producing a torque on the bicycle crank. The amount of this torque will vary, even if we push with a constant force. This is because the effective length of the crank acting as a lever arm is changing as the crank rotates. The torque will be at maximum when the force being exerted by our foot is perpendicular to the crank and the lever arm is therefore at its greatest length.

In Figures 10.3a and 10.3b, F is the applied constant force. The crank is of unvarying length. However, the length of the lever arm L changes as the crank rotates. We must determine the length of the lever arm, in this case the component of the crank that is perpendicular to the line of action of the force, in order to accurately calculate the turning force acting on the crank.

Torque is also used to describe engine and vehicle performance. An engine generates torque that twists on the transmission. The transmission makes the torque larger and transfers it to the differential. The differential further increases the torque through gearing and transfers the torque to the

drive axle. The twisting force of the drive axle needs to be transformed into translational force to make the car go forward. This is done by solving for force from the definition of torque:

$$F = \frac{\tau}{L} \tag{10.8}$$

where τ is the magnitude of the torque and L is the length of the lever arm. Notice that if the length of the lever arm is smaller, the force generated by the torque becomes larger. This means installing larger tires on a vehicle will *decrease* the force used for acceleration and *increase* the force required for braking.

Whenever a vehicle locks its wheels and skids, the torque generated by the brakes is greater than the torque generated by the road. Since the lever arm for a brake is the radius of the drum and the lever arm for the road is the rolling radius of the tire, the brake's frictional forces must be higher than the frictional forces of the tire/ground interface.

Let's say that we have a three-dimensional vector $\boldsymbol{\tau} = (\tau_x, \tau_y, \tau_z)$ that defines some torque. We also have a lever arm that is perpendicular to the torque vector. How can we find the force vector generated by this lever arm and torque?

There are many vectors that, when cross-multiplied, can give the desired torque vector. For this reason, the inverse of the cross product does not exist. However, if we make all vectors perpendicular to each other,* then there is a unique solution to the inverse cross product problem. The result without derivation is:

$$\mathbf{F} = \frac{\mathbf{L} \times \boldsymbol{\tau}}{\mathbf{L} \cdot \mathbf{L}} \tag{10.9}$$

where $(\times)$ indicates cross product and $(\cdot)$ indicates dot product.

10.7.1 Torque Example

Determine the force vector from the torque generated by a drive wheel. The axle shaft produces a torque of 3600 in-lb and the rolling radius is 15 inches.

The torque is originally defined as

$$\boldsymbol{\tau} = \mathbf{F} \times \mathbf{L}$$

*When vectors are mutually perpendicular, they are *orthogonal*.

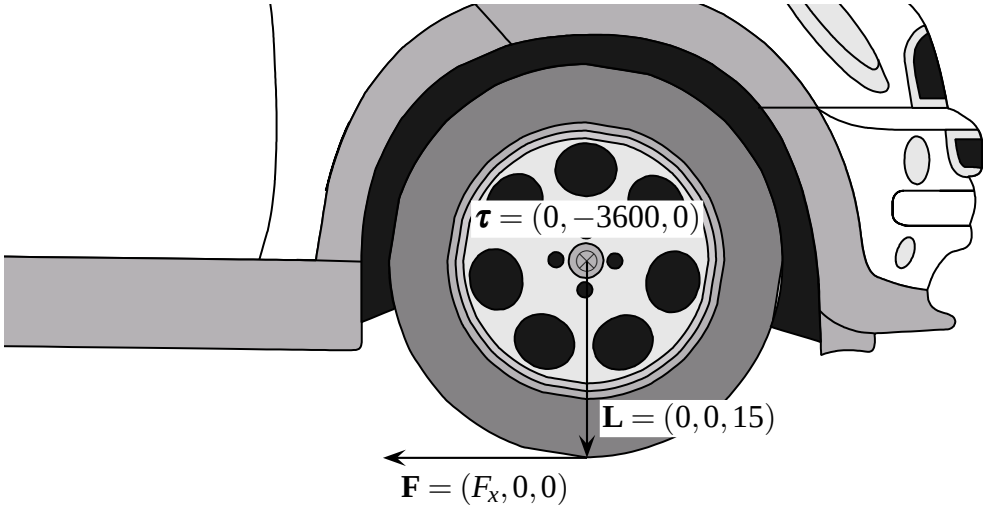


Figure 10.4: Determine F_x using Eq. (10.9). The torque vector from the axle shaft is going into the page based on the right hand rule. Positive directions are based on SAE standards.

Since we know that all three vectors are perpendicular, we can solve for **F** using Eq. (10.9):

$$\mathbf{F} = \frac{\mathbf{L} \times \boldsymbol{\tau}}{\mathbf{L} \cdot \mathbf{L}}$$

$$(F_x, 0, 0) = \frac{(0, 0, 15) \times (0, -3600, 0)}{(0, 0, 15) \cdot (0, 0, 15)}$$

$$(F_x, 0, 0) = \frac{(-54000, 0, 0)}{225}$$

This gives the result:

$$F_x = -240\,\text{lb}$$

Notice this magnitude is the same magnitude as determined by $F = \dfrac{\tau}{L}$. This is the force the road is placing on the tire to propel the vehicle forward.

10.8 The Concept of Equilibrium

Newton's three laws of motion will help us to understand the concept of equilibrium. An object is in equilibrium whenever all the forces add (in a vector sense) to zero. Since there is no net (unbalanced) force, the object will remain at rest or remain at a constant velocity according to

Newton's First Law. This means that if there is no net force, then there will be no acceleration. In other words, the object is not speeding up, slowing down, or changing direction.

For example, a book sitting still on a table is in equilibrium since there are no horizontal forces and there are no unbalanced vertical forces. This is due to Newton's Third Law that says for each action, there is an equal and opposite reaction; so, the table is pushing up on the book just as hard as the book is pushing down on the table. Another example of equilibrium is a passenger in a train while the train is moving down the tracks at a constant velocity. The passenger is in equilibrium with the train and the surroundings because there are no unbalanced forces acting on him or her.

We know that we can write all vectors as components in the x-direction and in the y-direction. This gives us the basis for writing the equations of equilibrium as a sum of all the force vector components being equal to zero:

$$\sum F_x = 0 \tag{10.10}$$

$$\sum F_y = 0 \tag{10.11}$$

For every problem that can be represented in a 2-D plane (drawn in 2-D) there are three degrees of freedom: translation in the x-direction, translation in the y-direction, and rotation. Therefore, there needs to be three equations for these three degrees of freedom. We have presented the two equilibrium equations for translation. To keep the body from rotating, we have to have all of the applied moments (torques) equal to zero.

$$\sum M_z = 0 \tag{10.12}$$

The subscript z is used because the direction of the moment vector points in the z-direction. The right hand rule and the definition of a cross product shows the z-axis to be coming out of the page.

10.8.1 Equillibrium Example

Consider an open tailgate of a pickup truck. It is supported by the hinge at the base of the tail gate and by steel cables connected at an angle. If a forklift sets an object with a mass of 1000 kg (2200 lb) on the center edge of the tailgate, how much tension is in each cable? Assume the weight of

the tailgate to be much less than the load. The height of the tailgate is 0.60 m (23.62 in) and the connection point is 0.3 m (11.81 in) from the hinge.

Solution

We begin by drawing a free body diagram of the tailgate, as shown in Figure 10.5 on page 10-13. We must also figure out how much force is applied by a 1000-kg mass. If we use Eq. (10.4) we remember that weight is a force determined by $w = mg$. In the US system, 1000 kg is equal to 68.5 slugs (see Table 5.2 on page 5-6). This is equivalent to 2200 lb. In the SI system, 1000 kilograms weighs 9800 newtons.

Figure 10.5 depicts two torques acting on the tailgate. The first is the torque produced by the load, w, multiplied by the height of the tailgate, l_w. This torque will cause the tailgate to rotate clockwise since the hinge is on the left. The second torque is produced by the two cables holding up the tailgate and load. Both cables are in tension and produce an overall torque that acts against the torque produced by the load sitting on the tailgate.

Since the cables are attached at an angle to the tailgate, we must determine the overall magnitude of the force produced by the cables acting perpendicular to the tailgate. This force can be expressed as $T \sin\theta$.

Since the tailgate is in equilibrium, the sum of the moments (torques) is equal to zero. If we sum all the torques about the hinge and set them equal to zero, we get:

$$0 = -l_w(w) + l_T\,(T \sin\theta)$$

We are looking for the sum of the total torques acting on the tailgate. The $-l_w(w)$ term is the total torque from the load and $T \sin\theta$ is the total torque from both cables. If we solve for T we get:

$$\begin{aligned} l_T\,(T \sin\theta) &= l_w(w) \\ T \sin\theta &= \frac{l_w(w)}{l_T} \\ T &= \frac{l_w(w)}{l_T \sin\theta} \end{aligned}$$

Making the appropriate substitutions:

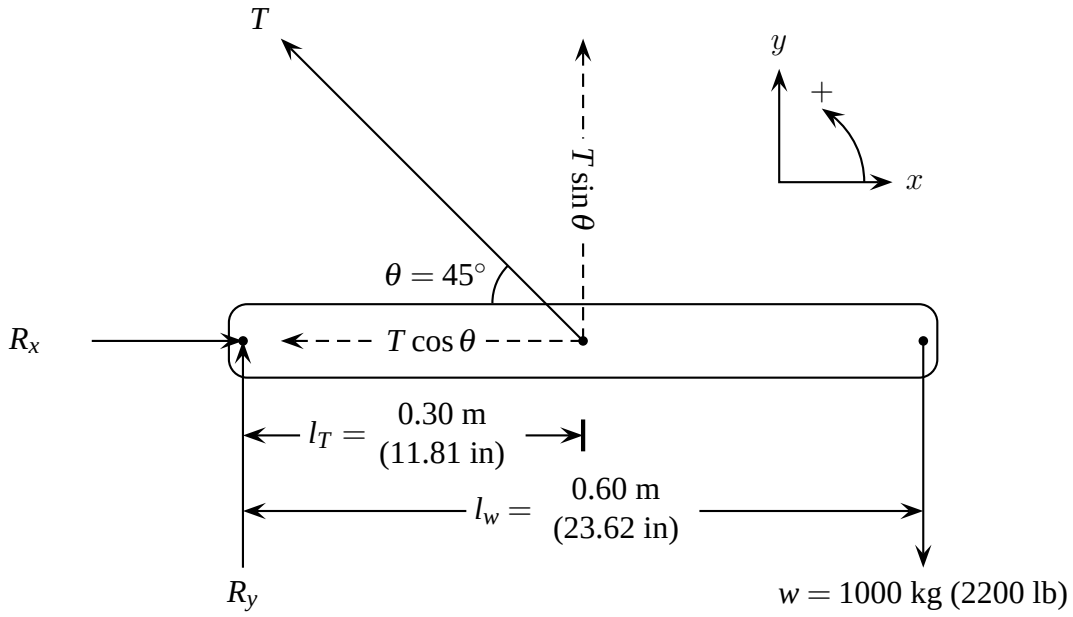


Figure 10.5: The free body diagram of a pickup truck tailgate supporting a load of 1000 kg (2200 lb) on its edge. There are two cables, each carrying half the load. The reaction force components of the hinge are labeled $R_x$and R_y.

US	SI
$T = \dfrac{23.62(2200)}{11.81(\sin 45^\circ)}$	$T = \dfrac{0.60(1000)}{0.30(\sin 45^\circ)}$
$= 6222.53\,\text{lb}$	$= 2828.42\,\text{kgf}$

Now divide by 2 to determine the tension in each cable:

$T_{cable} = \dfrac{T}{2}$	$T_{cable} = \dfrac{T}{2}$
$= \dfrac{6222.53}{2} = 3111.26\,\text{lb}$	$= \dfrac{2828.42}{2} = 1414.21\,\text{kgf}$

These are strong cables if they do not break! The tension in each cable is equal because the load

is centered between the cables. If the load were off-center, the tension in each cable would be different. The concept of off-center loading is explored in the next example.

Example

Determine the reaction forces holding the end of the bar that is loaded off-center, as shown in Figure 10.6. The weight of the bar is small compared to the 100-pound (445-N) weight. The bar is resting on two supports, labeled #1 and #2, that produce the reaction forces.

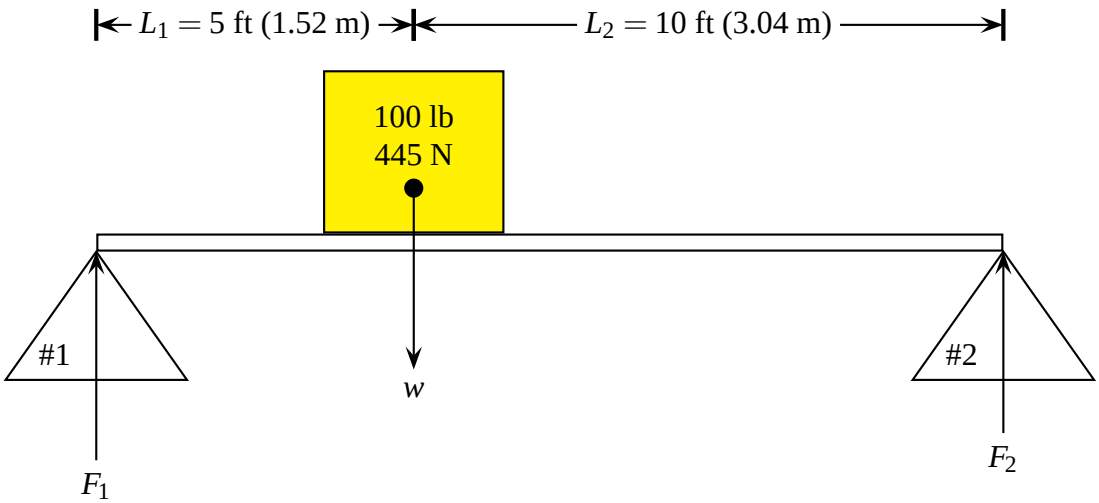


Figure 10.6: A very light-weight bar loaded off-center with a weight. The reaction forces F_1 and F_2 are determined using the concept of torque.

In Figure 10.6, F_1 is the upward force exerted by support #1, F_2 is the upward force exerted by support #2, L_1 is the length of the arm from the 100-pound (445-N) weight to support #1, and L_2 is the length of the arm from the 100-pound (445-N) weight to support #2.

We want to find the forces F_1 and F_2 exerted by the two supports. The bar is not moving, which means that all the forces and torques acting on the bar are in equilibrium. In other words, they all cancel each other out. Also, since the bar is not moving up or down, the downward force of 100 pounds (445 N), which is shared (unequally) by the two supports, is balanced by an upward force of 100 pounds (445 N). We can express this upward force by the following equation:

US		SI	
$F_1 + F_2 = 100\,\text{lb}$	(10.13)	$F_1 + F_2 = 445\,\text{N}$	(10.14)

Similarly, all of the torques must balance, since the bar is not rotating. To make our calculations easier, let's consider the pivot point for the lever arms to be the point on the bar where the 100-pound (445-N) force is acting. Since the torques cancel each other, they must be equal; so:

$\tau_1 = \tau_2$		$\tau_1 = \tau_2$	
$F_1 L_1 = F_2 L_2$	(10.15)	$F_1 L_1 = F_2 L_2$	(10.17)
$F_1(5) = F_2(10)$	(10.16)	$F_1(1.52) = F_2(3.04)$	(10.18)

where each side represents the torque produced by the reacting forces (torque equals force times length of lever arm.)

We will now rewrite Eqs. (10.13) and (10.14) for F_1. This gives us

US	SI
$F_1 = 100 - F_2$	$F_1 = 445 - F_2$

Next, we will substitute each expression for F_1 into Eq. (10.16) and Eq. (10.18) respectively:

US	SI
$(100 - F_2)(5) = F_2(10)$	$(445 - F_2)(1.52) = F_2(3.04)$

We then solve for F_2:

US	SI
$500 - 5F_2 = 10F_2$	$676.4 - 1.52F_2 = 3.04F_2$
$500 = 15F_2$	$676.4 = 4.56F_2$
$F_2 = 33.33\,\text{lb}$	$F_2 = 148.33\,\text{N}$

It is now a simple matter to substitute the value for F_2 into Eq. (10.13) or Eq. (10.14) to find F_1:

$F_1 + 33.33 = 100$	$F_1 + 148.33 = 445$
$F_1 = 66.67\,\text{lb}$	$F_1 = 296.67\,\text{N}$

Thus, F_1 and F_2 are the forces being exerted by supports #1 and #2, respectively, to keep the bar in equilibrium. Up to this point, we have treated the bar as weightless. If we now consider the bar itself to weigh 20 pounds (89 N) and to have the 20 pounds (89 N) uniformly distributed over its length, each support will then exert an additional 10 pounds (44.5 N) to keep the bar in a state of equilibrium. Unlike the 100-pound (445-N) load, the weight of the bar will not be concentrated off-center.

10.9 Center of Mass

All objects have a center of mass. The idea of a body having a center of mass is important because it allows us to simplify many of the analyses we perform. The definition of center of mass can be stated as:

> The center of mass, or CM, of a body is the point at which the entire weight of the body appears to be concentrated.

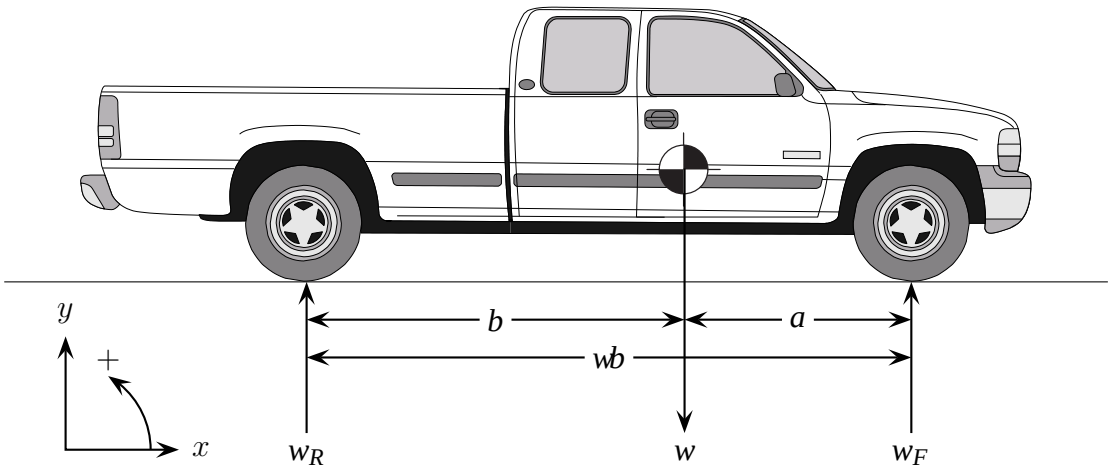


Figure 10.7: Determining the center of mass of a vehicle requires knowing the axle weights and wheel base. Solving the equations of equilibrium gives the distance, a, of the CM behind the front axle.

In a uniform gravitational field, the center of mass and center of gravity are the same point. Also, the centroid of a geometric figure or volume is analogous to a center of mass. Therefore, if an object is symmetric (like vehicles), then its center of mass is located along the line (or plane) of symmetry.

10.9.1 Determining Center of Mass

How do we actually determine the center of mass of a vehicle? The center of mass is the location in a body where all of the weight appears to be concentrated. This means that the point has to be located somewhere within the body. Physics (and logic) tells us that the center of gravity has to be between the wheels or else the car would tip up on its end. There are three pieces of information needed to determine the horizontal distance behind the front axle that the center of mass is located. This distance is denoted in Figure 10.7 as a.

1. The wheel base length,[†] $wb = a + b$
2. The weight on the front axle, w_F
3. The weight on the rear axle, w_R

We begin by drawing a free body diagram, as shown in Figure 10.7.

[†]There are three variables used here that may be confusing. Weight is denoted as w, the rear axle distance is b, and the wheelbase length is wb. The variable wb is not $w \times b$.

This problem is a static problem because the truck is not moving. We solve this by summing all the forces and moments and setting them equal to zero. There are no forces in the x-direction, so we only consider forces in y and moments (or torques) about some point.

The forces in the vertical direction come from the weight of the object. Newton's Third Law states that there is an equal and opposite reaction for every action, so the surface must push up on the truck with a sum total of the weight of the truck. These reaction forces are acting on the tires and are denoted as w_F and w_R for the front and rear axles, respectively. The sum of the vertical forces are zero, which is written as:

$$0 = w_F + w_R - w$$

so

$$w = w_F + w_R$$

The sum of the moments about the front tire has two terms. The first term is from the weight, w, of the vehicle acting through a lever arm of length a, and the second term is the reaction force at the rear wheel, w_R, acting with a lever arm of the length of the wheelbase wb. We see that by our sign convention, a positive moment is counter-clockwise. Therefore, the equilibrium equation for torques is:

$$0 = wa - w_R wb$$

Solving for the distance the center of mass is behind the front axle, a gives

$$a = \frac{w_R}{w}(wb)$$

which demonstrates that the ratio of the rear axle weight to the total weight is the ratio of the distance of the CM behind the front axle to the wheel base.

10.9.2 Center of Mass Height

The height of the center of mass off the ground can be determined by raising an axle off the ground while weighing the other axle. Figure 10.8 shows the rear end of a pickup raised while the front axle weight is measured. The angle the vehicle makes with the ground should be at least 15° to

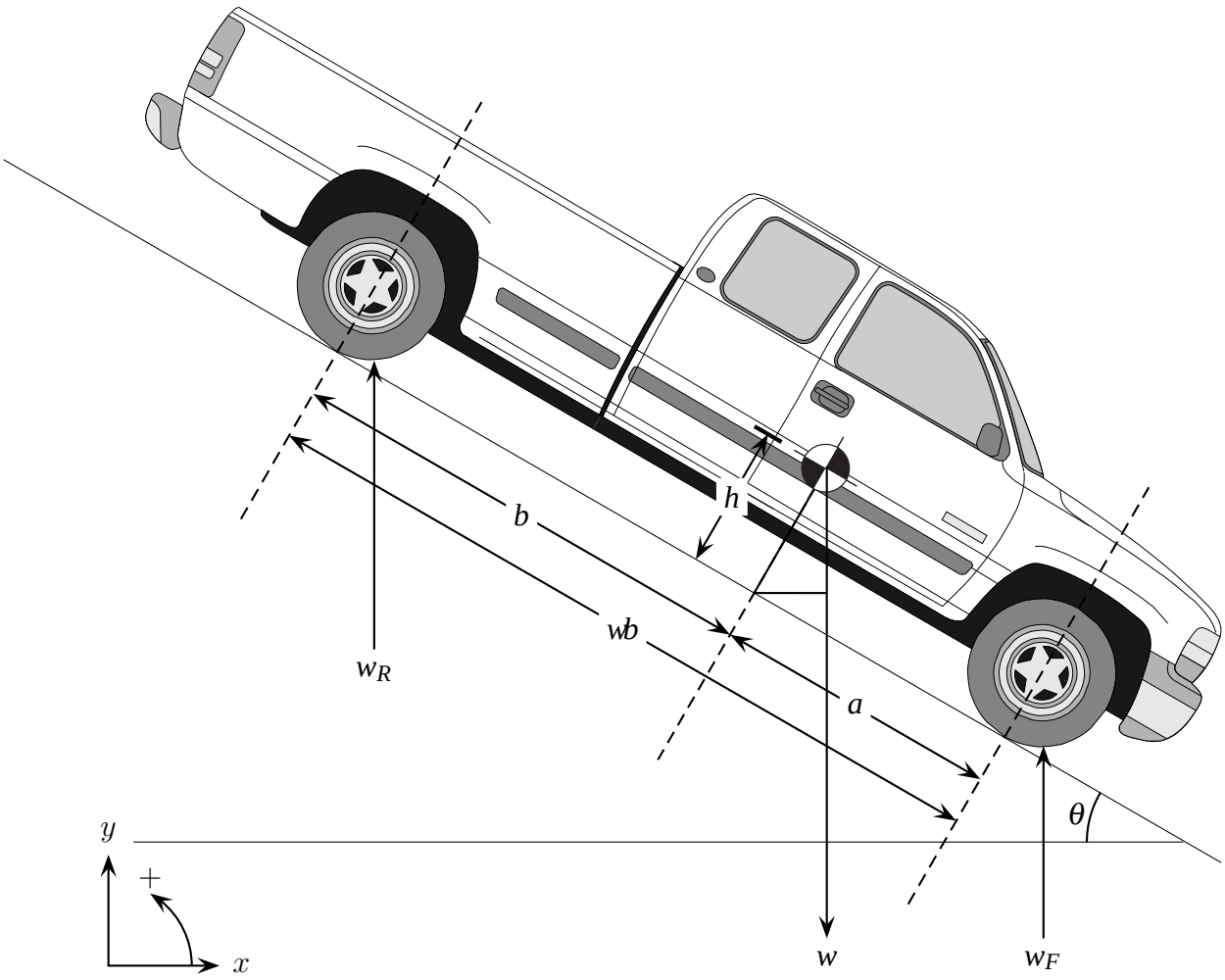


Figure 10.8: To measure the height of the center of mass, the vehicle should be at an angle.

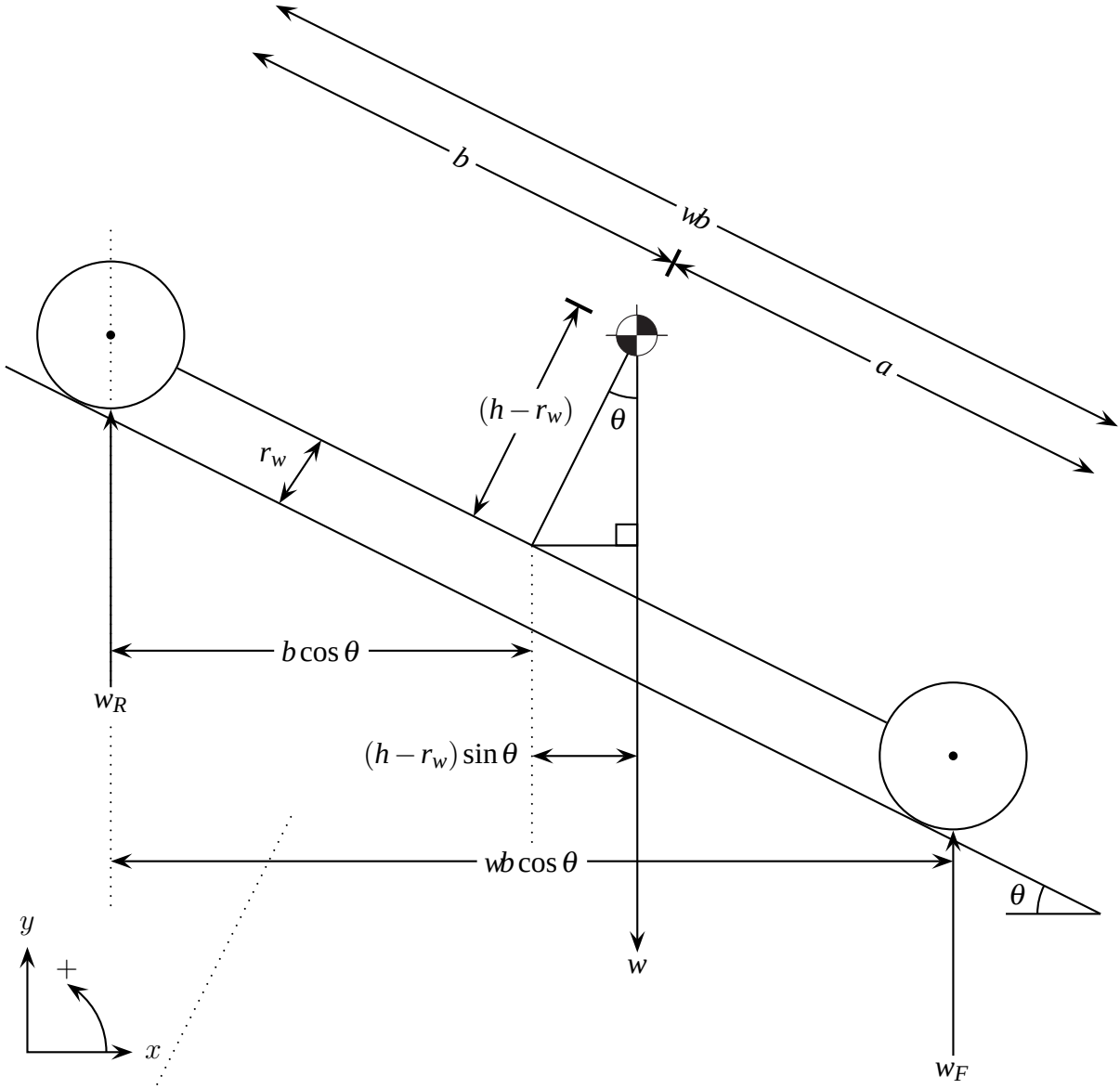


Figure 10.9: A free body diagram of the truck that shows pertinent forces and dimensions. The radius of the tires is r_w and the height of the center of mass from the ground plane is h.

ensure the variation in the scale does not propagate to the determination of the CM height. The minimum height the rear should be raised can be determined by using the sine function, for which the hypotenuse is the wheelbase and the height the rear axle is raised is the side opposite the angle:

$$h_{raised} = wb \sin\theta \tag{10.19}$$

Thus, in the field, the minimum height the rear axle should be raised can quickly be determined by multiplying the wheelbase of the vehicle by 0.258, which is the sine of 15°. The actual angle can be determined by solving Eq. (10.19) for θ.

$$\theta = \sin^{-1} \frac{h_{raised}}{wb} \tag{10.20}$$

Keep in mind that the suspension should be loaded (compressed) as it was on level ground when the height is measured. This means that the axle should be raised, not just the frame.

We already know the weight of the vehicle and its horizontal location of the CM from the previous section. These data along with the new front axle weight w_F and the angle of inclination will allow us to calculate the height of the center of mass by balancing the torques generated by the mass of the vehicle and the reaction forces at the front wheel.

We'll start this analysis with a free body diagram of the scenario, as shown in Figure 10.9. The sum of the moments about the rear axle is composed of two torques:

1. The total weight of the vehicle w acting through a moment arm of $b\cos\theta + (h - r_w)\sin\theta$. This is a negative moment.

2. The weight on the front axle w_F that acts through a lever arm of $wb\cos\theta$. This is a positive moment.

These moments sum to zero:

$$0 = w_F(wb\cos\theta) - w(b\cos\theta + (h - r_w)\sin\theta) \tag{10.21}$$

We are able to determine the angle θ from Eq. (10.20) and the total weight, wheel base, and rear distance b of the CM are known from the previous section, so the only unknown in Eq. (10.21) is

the height of the CM. Solving for h yields:

$$
\begin{aligned}
w_F(wb\cos\theta) &= w(b\cos\theta + (h - r_w)\sin\theta) \\
\frac{w_F(wb\cos\theta)}{w} &= b\cos\theta + (h - r_w)\sin\theta \\
\frac{w_F(wb\cos\theta)}{w} - b\cos\theta &= (h - r_w)\sin\theta \\
\left(\frac{w_F wb}{w} - b\right)\frac{\cos\theta}{\sin\theta} &= h - r_w \\
\left(\frac{w_F wb}{w} - b\right)\frac{1}{\tan\theta} &= h - r_w
\end{aligned}
$$

Rearranging gives:

$$\boxed{h = r_w + \frac{w_F w b}{w \tan\theta} - \frac{b}{\tan\theta}} \tag{10.22}$$

Keep in mind, this equation is valid for the configuration for which it was derived (i.e. Figure 10.8). A similar technique can be used to derive the center of mass height if other information is given.

10.9.3 Center of Mass with Cargo

If a pickup has a payload that was lost during a crash, how can we determine its center of mass? We need the following information:

1. The weight of the payload. This means all the debris from the back of a pickup becomes evidence.
2. The location of the center of mass of the payload while it is in the truck. Normally, this will be estimated based on witness statements.
3. The location of the center of mass of the truck.

Consider Figure 10.10 on page 10-24, which shows a pickup with a known center of mass location and its payload. The payload has a center of mass as shown. If we choose a reference point to be where the front tire makes contact with the road, we can determine the location of the combined center of mass using principles of static equilibrium.

The procedure for determining the combined center of mass follows that of determining the centroid of a geometric shape, as discussed in Section **??** on page ??. However, in this example, the weighting function is from the gross weights rather than the area. Thus:

$$x_{CM} = \frac{\sum x_i w_i}{\sum w_i} \tag{10.23}$$

$$y_{CM} = \frac{\sum y_i w_i}{\sum w_i} \tag{10.24}$$

where $\sum$ indicates summation, x_i is the x-distance from a reference point, y_i is the y-distance from the reference point, and w_i is the weight of the individual objects. If we sum all of the weights, then we have to total combined weight:

$$w_{total} = \sum w_i$$

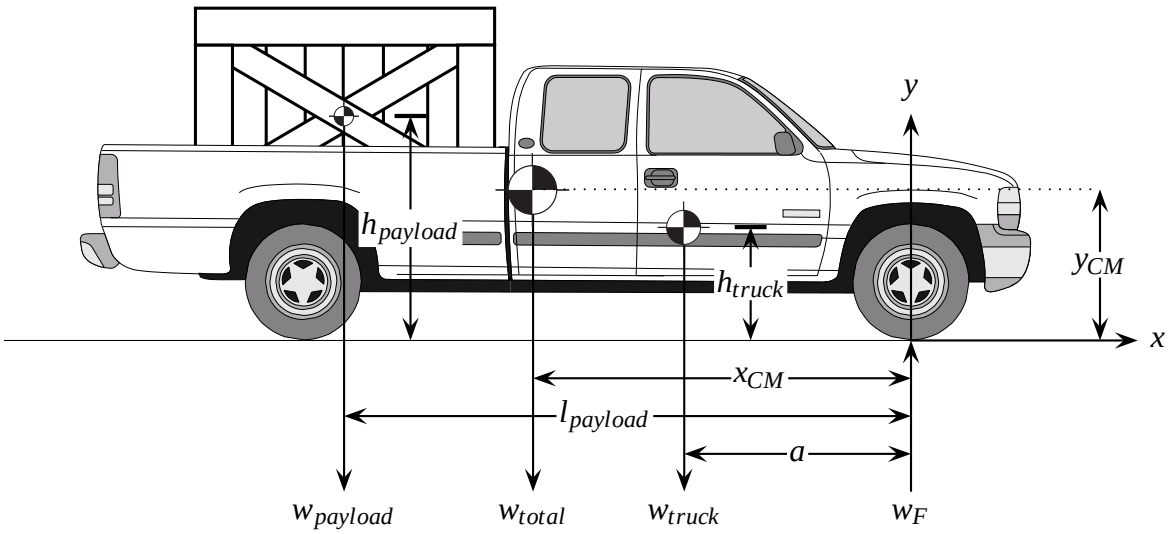


Figure 10.10: The location of the combined center of mass can be determined by the distances x_{CM} and y_{CM}, as measured from the front tire/ground interface.

10.9.4 Center of Mass with Cargo Example

Determine the center of mass location of the combined pickup and crate in Figure 10.10 when:

$w_{truck} = 4400$ lb (1995 kg) $\qquad$ $w_{payload} = 1500$ lb (680 kg)

$a = 36$ in (0.914 m) $\qquad$ $l_{payload} = 100$ in (2.54 m)

$h_{truck} = 24$ in (0.61 m) $\qquad$ $h_{payload} = 50$ in (1.27 m)

The total weight of the truck and its payload is:

US

$$\begin{aligned} w_{total} &= \sum w_i \\ &= w_{truck} + w_{payload} \\ &= 4400 + 1500 \\ &= 5900\,\text{lb} \end{aligned}$$

SI

$$\begin{aligned} w_{total} &= \sum w_i \\ &= w_{truck} + w_{payload} \\ &= 1995 + 680 \\ &= 2675\,\text{kg} \end{aligned}$$

The location of the center of mass in the x-direction is found using Eq. (10.23):

US

$$\begin{aligned} x_{CM} &= \frac{\sum x_i w_i}{\sum w_i} \\ &= \frac{-a(w_{truck}) - l_{payload}(w_{payload})}{w_{total}} \\ &= \frac{-36(4400) - 100(1500)}{5900} \\ &= -52.27\,\text{in} \end{aligned}$$

SI

$$\begin{aligned} x_{CM} &= \frac{\sum x_i w_i}{\sum w_i} \\ &= \frac{-a(w_{truck}) - l_{payload}(w_{payload})}{w_{total}} \\ &= \frac{-0.914(1995) - 2.54(680)}{2675} \\ &= -1.32\,\text{m} \end{aligned}$$

The minus sign indicates that the combined center of mass is located to the left of the y-axis (behind the front axle).

The location of the center of mass in the y-direction is found using Eq. (10.24):

$$
\begin{aligned}
y_{CM} &= \frac{\sum y_i w_i}{\sum w_i} \\
&= \frac{h_{truck}(w_{truck}) + h_{payload}(w_{payload})}{w_{total}} \\
&= \frac{24(4400) + 50(1500)}{5900} \\
&= 30.61\,\text{in}
\end{aligned}
$$

$$
\begin{aligned}
y_{CM} &= \frac{\sum y_i w_i}{\sum w_i} \\
&= \frac{h_{truck}(w_{truck}) + h_{payload}(w_{payload})}{w_{total}} \\
&= \frac{0.61(1995) + 1.27(680)}{2675} \\
&= 0.778\,\text{m}
\end{aligned}
$$

Thus, the center of mass of the pickup and payload combination is 52.27 in (1.32 m) behind the front axle and 30.61 in (0.778 m) above the ground.

10.10 Dynamic Weight Shift

When a vehicle accelerates, force is transfered from one axle to another. In the extreme case, a motorcycle will accelerate quickly enough to transfer all the weight to the rear wheel and do a "wheelie." Similarly, when a car brakes suddenly, the nose of the car pitches down and "dives." These phenomena are explained with dynamic weight shift.

To understand the concept of weight shift, we must first introduce a concept called *inertial* force. When an object is acted on by external, unbalanced forces, the object accelerates at a rate inversely proportional to its mass. This is a statement of Newton's Second Law:

$$F = ma \tag{10.25}$$

If we subtract ma from both sides

$$
\begin{aligned}
F - ma &= ma - ma \\
F - ma &= 0
\end{aligned} \tag{10.26}
$$

Now Eq. (10.26) has the same form as an equilibrium problem in which the sum of the forces are equal to zero. For this reason, we can call the quantity ma a *pseudo* force and use it in our force summing equations defined by Eqs. (10.10-10.11). This pseudo force is a result of the concept

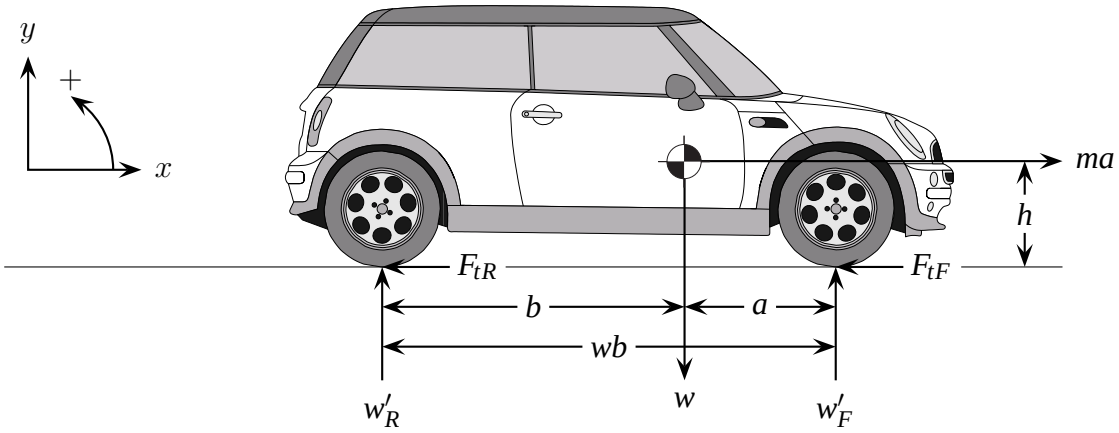


Figure 10.11: A braking vehicle (acceleration is negative) with an inertial force acting at the center of mass in the opposite direction of the acceleration. Positive direction is defined as the direction of the velocity vector.

of inertia and the term *inertial force* is often used to describe the amount of force being used to accelerate an object. The inertial force always acts in the *opposite* direction of the acceleration, so the negative sign must be included when summing the forces to use in Eqs. (10.10-10.11). The inertial force always acts from the center of mass and can generate a moment if a lever arm exists, as in the case of a vehicle braking.

Let us now solve the problem for dynamic weight shift of a braking vehicle. Consider a vehicle slowing down, as in Figure 10.11. This vehicle is actually accelerating in the negative direction so the inertial force is acting in the positive direction. The external forces from the road act in two components: normal and tangential (parallel). The normal force is the force perpendicular to the road. The tangential force is due to the friction between the road and the tire when the vehicle is level. If the coefficients of friction μ for each axle are known, then the tangential forces are:

$$F_{tF} = \mu_F w'_F \tag{10.27}$$

$$F_{tR} = \mu_R w'_R \tag{10.28}$$

where w'_F is the dynamic weight on the front axle and w'_R is the dynamic weight on the rear axle when the vehicle is level.

We can balance the forces in the x-direction (parallel to the road):

$$ma = F_{tF} + F_{tR} \tag{10.29}$$

$$ma = \mu_F w'_F + \mu_R w'_R \tag{10.30}$$

Similarly, we can balance forces in the y- direction:

$$w = w'_F + w'_R \tag{10.31}$$

Finally, we sum the moments about the rear contact patch of the vehicle in Figure 10.11.[‡] We see that there are three moments:

1. The weight of the vehicle w with a lever arm of b.
2. The front reaction force w'_F with a lever arm of wb.
3. The inertial force ma with a lever arm of h.

The sum of these moments is equal to zero.

$$0 = -b(w) + wb(w'_F) - h(ma) \tag{10.32}$$

Let us define a quantity of dynamic weight shift Δw_D. When a vehicle is not accelerating, the weight distribution on the front and rear axles is determined using the technique explained in Section 10.9. However, if the vehicle is decelerating, the weight distribution changes such that:

$$w'_F = w_F + \Delta w_D \tag{10.33}$$

$$w'_R = w_R - \Delta w_D \tag{10.34}$$

where w'_F is the new weight on the front axle due to weight shift and w'_R is the new weight on the rear axle due to weight shift. Notice that the overall weight does not change, only the distribution of the weight between the front and rear.

To determine Δw_D, we substitute Eqs. (10.30), (10.33), and (10.34) into Eq. (10.32):

$$0 = -bw + wb(w_F + \Delta w_D) - h(\mu_F[w_F + \Delta w_D] + \mu_R[w_R - \Delta w_D])$$

[‡]We can sum the moments about any point we desire and the results will be the same. The rear contact patch is chosen because it cancels out some moments.

Expanding all terms gives:

$$0 = -wb + wbw_F + wb\Delta w_D - h\mu_F w_F - h\mu_F \Delta w_D - h\mu_R w_R + h\mu_R \Delta w_D$$

From Section 10.9 we recall:

$$w_F = \frac{b}{wb} w$$

If we substitute this expression for the first occurrence of w_F, then the first two terms will cancel:

$$0 = -wb + \cancel{wb}\left(\frac{b}{\cancel{wb}} w\right) + wb\Delta w_D - h\mu_F w_F - h\mu_F \Delta w_D - h\mu_R w_R + h\mu_R \Delta w_D$$
$$0 = wb\Delta w_D - h\mu_F w_F - h\mu_F \Delta w_D - h\mu_R w_R + h\mu_R \Delta w_D$$

Isolate Δw_D :

$$h\mu_F w_F + h\mu_R w_R = \Delta w_D (wb - h\mu_F + h\mu_R)$$

Factor out h and solve for Δw_D:

$$\boxed{\Delta w_D = \frac{h(\mu_F w_F + \mu_R w_R)}{wb + h(\mu_R - \mu_F)}} \tag{10.35}$$

There is a simplified case for Eq. (10.35). This is when the coefficient of friction is the same for both axles $\mu = \mu_R = \mu_F$:

$$\Delta w_D = \frac{\mu h (w_F + w_R)}{wb}$$
$$\Delta w_D = \frac{\mu h w}{wb} \tag{10.36}$$

If the acceleration is positive, the vehicle is speeding up and the weight shift Δw_D is negative, so more weight is added to the rear axle. If the acceleration is negative, the vehicle is slowing down and the weight is shifted toward the front axle.

Exercises

1. An external unbalanced force of 440 newtons is applied to a box of 100 kg. What is the acceleration of the box

4.4 a) in meters per second per second?

0.44 b) in g's?

2. A mechanic is trying to remove a stubborn bolt. The bolt requires 700 ft-lb of torque to remove it.

933.3 lb a) If the wrench being used is 9 inches (0.75 ft), how much force is required to turn the bolt?

218.75 lb b) If a breaker bar of 3.2 ft is used to move the bolt, how much force is required to turn the bolt?

3. What torque must the drive axle generate to keep a vehicle from accelerating down a 20° slope if the rolling radius of the tires is 15 inches and the vehicle weighs 3000 lb?

15,390 in-lb

4. A trailer has a tongue weight of 300 lb when level and a total weight of 3600 lb. The trailer hitch is 109 inches in front of the single trailer axle. Where is the center of mass of the trailer with respect to the tongue?

99.91 in

5. As the tongue of the trailer in Exercise 4 is raised, it becomes lighter due to the height of the center of mass. If the load on the tongue disappears when the trailer is tilted up 30°, then what is the height of the center of mass of the trailer off the ground when the tire radius is 12 inches?

27.74 in

6. The trailer in Exercise 4 is placed on the hitch ball of a pickup. The unloaded pickup weighs 4800 lb and has a wheel base of 152 inches. When unloaded, the pickup has a weight distribution of 60% front and 40% rear. The hitch ball is located 58 inches behind the rear axle.

8400 lb a) What is the total force the ground is pushing up on the truck/trailer combination?

3300 lb b) How much weight is on the trailer tires?

2334 lb c) How much weight is on the rear tires of the truck?

2766 lb d) How much weight is on the front tires of the truck?

in

e) How far is the combined center of mass behind the front axle of the truck?

f) The height of the center of mass of the unloaded truck is 38 inches. Using the answer from Exercise 5, how high is the combined center of mass? 33.6 in

7. As a motorcyclist accelerates on a level surface, the front wheel comes off the ground. The wheel base of the motorcycle is 2.4 meters and the center of mass (with rider) is in the middle of the wheel base and 1.5 meters off the ground. The total mass of the system is 500 kg.

   a) What coefficient of friction is required for this maneuver to occur? 0.80

   b) If the coefficient of friction is only 0.60, will the front wheel come off the ground if the center of mass does not change? no

   c) What is the height of the center of mass required to raise the front wheel off the ground if the coefficient of friction is 0.60? (Assume the horizontal location of the CM remains in the middle of the wheel base.) 2 m

8. A 2600-lb sports car traveling at 80 mph accidentally down-shifts to 1st gear, which effectively locks the rear wheels. The maneuver occurs on a level surface with a coefficient of friction of 0.75 and the rolling resistance of the front wheels is $\mu = 0.02$. The height of the center of mass is 18 inches and the wheel base is 100 inches. The weight is evenly distributed between the front and rear. What is the dynamic weight shift? 159 lb

11 Linear Momentum

11.1 Linear Momentum and Impulse

Linear momentum is a quantity of motion that is very useful for crash investigators. Sir Issac Newton recognized momentum as a fundamental quantity of motion. In fact, Newton's laws of motion are used to derive the principles of momentum and conservation of momentum. Newton's First Law is also known as the Law of Inertia and states in part, "An object in motion tends to stay in motion unless acted upon by an external, unbalanced force." Momentum can be thought of as the tendency of a body to want to stay in motion; that is, momentum is inertia in motion.

A relationship that relates the changing of an object's momentum to the force that causes the change manifests itself as Newton's Second Law:

$$\mathbf{F} = m\mathbf{a}$$

where $\mathbf{F}$ is the net (unbalanced) force vector on an object, m is the mass of an object, and $\mathbf{a}$ is the acceleration vector. Recall that the average acceleration is defined as the change in velocity over a period of time, $a = \frac{\blacksquare\mathbf{v}}{\Delta t}$. Rewriting Newton's Second Law in terms of velocity and time yields:

$$\mathbf{F} = m\frac{\blacksquare\mathbf{v}}{\Delta t} \tag{11.1}$$

Multiplying both sides of this equation by Δt gives the relationship:

$$\mathbf{F}\Delta t = m\blacksquare\mathbf{v} \tag{11.2}$$

where the product of force and time on the left-hand side of Eq. (11.2) is the *impulse* and the right-hand side is a *change in momentum*. It is this impulse, this force applied over time, that changes an object's momentum. This is the net unbalanced force that Newton's First Law describes. Since the

right-hand side is a *change* in momentum, we can write an equivalent expression as:

$$\mathbf{F}\Delta t = m(\mathbf{v}_f - \mathbf{v}_o)$$
$$\mathbf{F}\Delta t = m\mathbf{v}_f - m\mathbf{v}_o$$

impulse = final momentum – initial momentum

where $\mathbf{v}_f$ is the final velocity vector and $\mathbf{v}_o$ is the initial velocity vector. We have thus defined the fundamental quantity of motion known as momentum, which is the product of mass and velocity.

Impulse is the product of force and time, $\mathbf{F}\Delta t$. In an impact, time intervals are typically short, which results in a high force for a given impulse. Impulse is a vector quantity because it is the product of a scalar (time) and a vector (force).

Momentum is the product of mass and velocity and is denoted with a **P** such that:

$$\mathbf{P} = m\mathbf{v} \tag{11.3}$$

Momentum is the tendency of a body to continue moving at a constant velocity. It is important to realize that momentum is a vector quantity because it the product of a scalar (mass) and a vector (velocity), and momentum has both magnitude and direction. The magnitude of momentum is $P = mv$.

Momentum has the units of mass times velocity, so any units of mass and velocity are acceptable to describe momentum. Similarly, impulse has the units of force times time. Equation (11.2) demonstrates that impulse is equal to momentum change, so the units for both impulse and momentum must be the same. A dimensional analysis (Chapter 5) will show that a slug-ft/sec (for momentum) is also equal to a lb-sec (for impulse) in the US system. Similarly, a kg-m/sec is the same as a newton-sec in the SI system. Therefore, the fundamental units of impulse are also mass times velocity, and units of impulse (force times time) can be used to describe quantities of momentum.

Example 11.1. A 6000-lb (2721-kg) Suburban slams into a giant oak tree from 45 mph (72 kph) and does not rebound.

1. How much momentum does the Suburban possess while traveling 45 mph (72 kph)?
2. What is the magnitude of the impulse delivered to the tree?
3. If the crash transpired in 110 ms, what was the average force acting on the Suburban?

Solution

1. The impulse can be calculated from a change in momentum. Since the vehicle did not rebound after striking the tree, then the final momentum is zero. This initial momentum is calculated with Eq. (11.3). Only the direction associated with the impact angle of the vehicle is present, so we will deal with magnitude only.

US	SI
$P = mv_o$	$P = mv_o$
$= (6000)(45)$	$= (2721)(72)$
$= 270,000\,\text{lbm-mph}$	$= 195,912\,\text{kg-kph}$

The units on these momentum values are correct, but not very useful for solving the problem at hand. Therefore, let's obtain the initial momentum in terms of fundamental units (i.e., slugs, feet, and seconds for the US system, and kilograms, meters, and seconds for the SI system).

US	SI
$m = \dfrac{w}{g} = \dfrac{6000}{32.2} = 186.33\,\text{slugs}$	$m = 2721\,\text{kg}$

Determine velocity in fundamental units:

$v_o = 1.466S$	$v_o = \dfrac{S}{3.6}$
$= 1.466(45)$	$= \dfrac{72}{3.6}$
$= 65.97\,\text{ft/sec}$	$= 20\,\text{m/sec}$

US	SI
$P = mv_o$	$P = mv_o$
$= (186.33)(65.97)$	$= (2721)(20)$
$= 12,292.19\,\text{slug-ft/sec}$	$= 54,420\,\text{kg-m/sec}$

2. Since a change in momentum is equal to the impulse, as shown in Eq. (11.2), the impulse delivered to the tree is:

US	SI
$\text{impulse} = 12,292.19\,\text{lb-sec}$	$\text{impulse} = 54,420\,\text{N-sec}$

   It is worthwhile to see that the units of impulse are the same as the units of momentum.

3. The average force on the Suburban is exerted by the tree and is equal and opposite to the average force exerted by the Suburban on the tree. This is a restatement of Newton's Third Law. So, the magnitudes of the impulsive forces are the same. Therefore, by definition of impulse from Eq. 11.1, we can calculate average force as follows:

$$F = \frac{m\Delta v}{\Delta t} = \frac{\text{impulse}}{\Delta t}$$

US	SI
$F = \dfrac{12,292.19}{0.110} = 111,747\,\text{lb}$	$F = \dfrac{54,420}{0.110} = 494,727\,\text{N}$

   Notice that the units of force come out as expected because we obtained values of momentum based on fundamental units.

In the early history of physics, the concepts of momentum and energy were often confused. Momentum was defined then as "a quantity of motion." As you might expect, this led to confusion because kinetic energy is also a "quantity of motion."

The difference between energy and momentum is how the action of a certain force is carried out. If the force acts over a certain distance, we have a quantity of *energy.* However, if the force acts over a period of time, we have a quantity of *momentum.*

Example 11.2. A 60,000-lb (27,215-kg) tractor-trailer is traveling at 5mph (8 kph).

1. How much momentum does the tractor-trailer possess?
2. How fast would a 600-lb (272-kg) motorcycle have to be going in order to have the same momentum as the tractor-trailer?
3. How much kinetic energy does the tractor-trailer have?
4. At what speed must the motorcycle be travelling to equal the kinetic energy of the tractor-trailer?

Solution

1. Momentum is calculated with Eq. (11.3).

US	SI
$P = mv$	$P = mv$
$= 60,000(5)$	$= 27,125(8)$
$= 300,000\,\text{lbm-mph}$	$= 217,000\,\text{kg-kph}$

2. The speed of the motorcycle can be calculated again from Eq. (11.3).

US	SI
$v = \dfrac{P}{m}$	$v = \dfrac{P}{m}$
$= \dfrac{300,000\,\text{lbm-mph}}{600\,\text{lbm}}$	$= \dfrac{217,000\,\text{kg-kph}}{272\,\text{kg}}$
$= 500\,\text{mph}$	$= 797.79\,\text{kph}$

Obviously, motorcycles cannot go this fast. In theory, a small, fast object can have the same momentum as a large, slow object.

3. The kinetic energy of the tractor-trailer is determined from Eq. (13.4) or (13.6):

US	SI
$KE = \frac{wS^2}{30}$	$KE = \frac{mS^2}{26}$
$= \frac{60,000(5)^2}{30}$	$= \frac{27,215(8)^2}{26}$
$= 50,000\,\text{ft-lb}$	$= 66,990\,\text{J}$

4. The speed required for the motorcycle to match the kinetic energy of the tractor-trailer is determined from Eq. (13.5) or (13.7):

US	SI
$S = \sqrt{\frac{30KE}{w}}$	$S = \sqrt{\frac{26KE}{m}}$
$= \sqrt{\frac{30(50,000)}{600}}$	$= \sqrt{\frac{26(66,990)}{272}}$
$= 50\,\text{mph}$	$= 80.02\,\text{kph}$

Since energy goes as speed squared, it is easy for a smaller, faster object to have more energy than a larger, slower object. In the above example, a small, fast motorcycle may have more energy than a slow tractor-trailer, but it is difficult for the motorcycle to have more momentum than the tractor-trailer.

Impulse and momentum also describe other phenomena. For example, if we hit a nail with a hammer, it is the effect of the changing momentum that imparts the force on the nail. Right before the hammer hits the nail, the hammer has a velocity toward the nail. The impact with the nail changes the velocity of the hammer. If this change in velocity is multiplied by the mass of the hammer, the change in momentum of the hammer is obtained. From Eq. (11.2) we see that this change in momentum is equal to the force of the hammer on the nail multiplied by the time of

impact. Since the impact duration is very small, the impact force is quite high when compared to the weight of the hammer.

11.2 Conservation of Linear Momentum

In a collision between two objects, collision times are typically short enough so that only the collision forces have enough magnitude to be considered. Therefore, the presence of external forces is negligible for most collisions. After making this assumption, we can invoke Newton's Third Law which says that every force has an equal and opposite force.

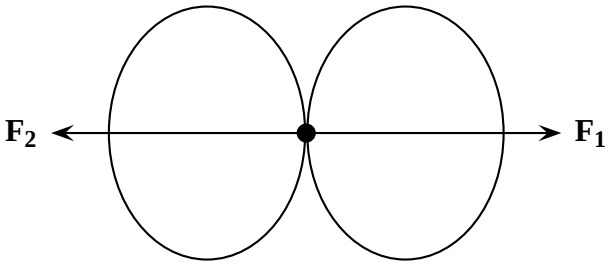


Figure 11.1: A collision between two objects produce equal and opposite forces.

$$\mathbf{F_1} = -\mathbf{F_2} \tag{11.4}$$

Since these collision forces are vectors, their magnitudes are the same but they act in opposite directions as shown in Figure 11.1. Furthermore, when two objects collide, the force application occurs over some period of time. That time is common to both forces which means the impulse vectors are equal and opposite:

$$\mathbf{F_1}\Delta t = -\mathbf{F_2}\Delta t \tag{11.5}$$

We saw in Eq. (11.2) that a change in momentum is equal to the impulse, $\mathbf{F}\Delta t = m\Delta\mathbf{v}$. However, if the change in velocity is defined such that $\blacksquare\mathbf{v} = \mathbf{v_o} - \mathbf{v_f}$, then $\Delta t = t_o - t_f$. Therefore, Δt is negative because the final time is always greater than the initial time. This definition, along with Eq. (11.2), leads to the observation that:

$$-\mathbf{F}\Delta t = m\Delta\mathbf{v} \tag{11.6}$$

We can substitute Eq. (11.6) into Eq. (11.5) to get

$$-m_1\blacksquare\mathbf{v_1} = m_2\blacksquare\mathbf{v_2} \tag{11.7}$$

which says that the change in momentum in one object is equal and opposite to the change in momentum of the other object. Eq. (11.7) can be expanded as:

$$-m_1(\mathbf{v_1} - \mathbf{v_3}) = m_2(\mathbf{v_2} - \mathbf{v_4}) \tag{11.8}$$

where m_1 is the mass of unit #1,
m_2 is the mass of unit #2,
$\mathbf{v_1}$ is the pre-impact velocity vector of unit #1,
$\mathbf{v_3}$ is the post-impact velocity vector of unit #1,
$\mathbf{v_2}$ is the pre-impact velocity vector of unit #2, and
$\mathbf{v_4}$ is the post-impact velocity vector of unit #2.

Equation (11.8) can be rearranged to equate the momentum before the impact to the momentum after the impact to arrive at the principle of *Conservation of Linear Momentum* (COLM):

$$\boxed{m_1\mathbf{v_1} + m_2\mathbf{v_2} = m_1\mathbf{v_3} + m_2\mathbf{v_4}} \tag{11.9}$$

This principle states that the total, or vector sum of, linear momentum of two or more bodies after a collision is the same as the total, or vector sum of, linear momentum possessed by the bodies before the collision.

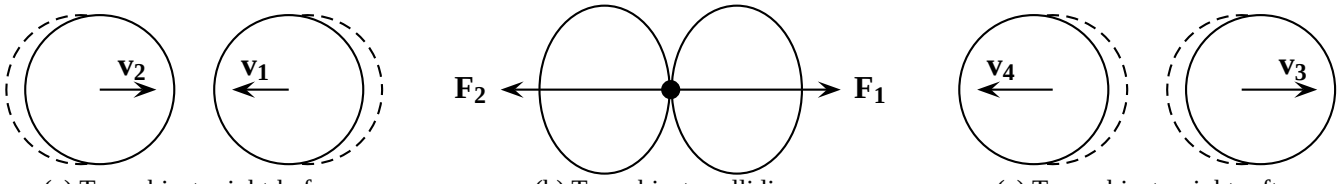
Eq. (11.9) is for two colliding bodies. Conservation of linear momentum can be extended to multiple objects and can be stated like this:

$$\boxed{m_1\mathbf{v_1}_{in} + m_2\mathbf{v_2}_{in} + \cdots + m_n\mathbf{v_n}_{in} = m_1\mathbf{v_1}_{out} + m_2\mathbf{v_2}_{out} + \cdots + m_p\mathbf{v_p}_{out}} \tag{11.10}$$

where n is the number of objects coming into the collision and p is the number of objects going out of the collision. The values of n and p do not have to be the same. A common example of this would be the break at the beginning of a pool game.

Keep in mind that all velocities (and thus momenta) are vector quantities, so direction matters. Also, Eq. (11.10) is the most general form of conservation of linear momentum when there are multiple objects and no external forces. Applications of this principle as it pertains to collision analysis are discussed in Chapter 12.

As mentioned at the beginning of this section, conservation of linear momentum was derived from the presumption that only collision forces exist during the collision and there are no outside influences on either body during the collision. Collisions involving land vehicles are not exempt from outside influence because the tires are exerting forces during the collision. However, if the collision forces are much higher than the tire forces, then the tire forces can be neglected without significantly changing the answer.



(a) Two objects right before a collision.

(b) Two objects colliding.

(c) Two objects right after the collision.

Figure 11.2: An in-line collision between two objects. The collision forces are always equal and opposite according to Newton's Third Law. Notice that only collision forces are present in the diagram.

11.3 In-Line Momentum

Let us consider only collisions that occur in a straight line, as depicted in Figure 11.2. We may refer to such collisions as being *collinear.* Conservation of linear momentum states that the momentum of the system is the same both before and after the collision. We know there are two objects involved so we can start with Eq. (11.9), which states that the total momentum coming into a collision is equal to the total momentum leaving the collision:

$$m_1\mathbf{v_1} + m_2\mathbf{v_2} = m_1\mathbf{v_3} + m_2\mathbf{v_4}$$

Since we are dealing with momentum, the direction of the velocity vector is determined by its sign: $(+)$ or $(-)$. We will introduce the term *direction cosine*[*] as the lower case Greek letter δ (lower case delta). The value of δ will either be $+1$ or -1, depending on the direction of the velocity vector. Furthermore, we can multiply the direction cosines by their corresponding velocity magnitudes to represent each velocity vector:

$$\mathbf{v_1} = \delta_1 v_1$$
$$\mathbf{v_2} = \delta_2 v_2$$
$$\mathbf{v_3} = \delta_3 v_3$$
$$\mathbf{v_4} = \delta_4 v_4$$

[*] As we are dealing with an in-line situation, the motion will be described along the *x*-axis. In Section **??**, we saw that the trigonometric function *cosine* was associated with the *x*-axis, hence the name "direction *cosine*."

We now take the principle of conservation of momentum, as shown in Eq. (11.9), and substitute the above relationships to get:

$$\delta_1 m_1 v_1 + \delta_2 m_2 v_2 = \delta_3 m_1 v_3 + \delta_4 m_2 v_4 \tag{11.11}$$

Equation (11.11) is a vector equation because each term has a magnitude (mv) and a direction δ. When considering traffic crashes, the direction cosines come from interpreting the physical evidence. The exit velocity magnitudes are either known or come from an energy based approach that is covered in Chapter **??**.

We can use an object's weight instead of its mass when applying Eq. (11.11). To see this, substitute $\frac{w}{g}$ for each occurrence of m in Eq. (11.11):

$$\delta_1 \left(\frac{w_1}{g}\right) v_1 + \delta_2 \left(\frac{w_2}{g}\right) v_2 = \delta_4 \left(\frac{w_1}{g}\right) v_3 + \delta_4 \left(\frac{w_2}{g}\right) v_4$$

Each term has a g in the denominator, so we can multiply through by g to get an in-line momentum equation in terms of weights:

$$\delta_1 w_1 v_1 + \delta_2 w_2 v_2 = \delta_3 w_1 v_3 + \delta_4 w_2 v_4 \tag{11.12}$$

Similarly, the units of the velocity magnitude can be in any valid unit for velocity as long as we are consistent. Normally, it is easier to work in units of miles per hour (kilometers per hour) rather than feet per second (meters per second).

Now let's apply Eq. (11.11) to a couple of examples.

Example 11.3. A ballistic pendulum is a large object that can absorb a fired bullet. It usually hangs by a couple of strings and has no initial velocity. A bullet is fired into the pendulum and the pendulum is knocked back at a measurable velocity. This velocity is measured by determining the height gained by the pendulum on its back swing. Let's say that a 1.0-oz (0.0283-kg) shotgun slug hits a 20-lb (9.07-kg) pendulum and sticks. The pendulum gains 0.50 ft (0.152 m) on the back swing. Determine the velocity of the slug.

Solution

The solution involves two steps. The first step comes from energy considerations found in Section 13.2 on page 13-6 in which we determine the initial velocity the pendulum needed to gain

the measured height. The pendulum has kinetic energy due to its *post* impact velocity. All of this energy is converted into potential energy when it reaches it maximum height. Keep in mind that the kinetic energy of the slug is *not* all converted to the kinetic energy of the pendulum. This conservation of energy is expressed as:

$$\begin{aligned} KE &= PE \\ \frac{1}{2}mv^2 &= mgh \end{aligned} \tag{11.13}$$

We can solve Eq. (11.13) for v. Notice that the mass cancels:

$$\begin{aligned} v^2 &= 2gh \\ v &= \sqrt{2gh} \end{aligned} \tag{11.14}$$

where v is the initial post impact velocity of the pendulum, g is the acceleration due to gravity, and h is the height gained by the pendulum on its back swing. Substituting the appropriate values into Eq. (11.14) gives:

US	SI
$v = \sqrt{2(32.2)(0.5)}$	$v = \sqrt{2(9.81)(0.1524)}$
$= 5.67\,\text{ft/sec}$	$= 1.728\,\text{m/sec}$

The second step is to solve the conservation of linear momentum Eq. (11.11):

$$\delta_1 m_1 v_1 + \delta_2 m_2 v_2 = \delta_4 m_1 v_3 + \delta_4 m_2 v_4$$

We can simplify this equation by realizing that all direction cosines are the same so $\delta_i = 1$. Also, the pendulum is initially at rest, so $v_2 = 0$. Since the bullet lodges inside the pendulum, the post impact velocities are the same for both masses, $v_3 = v_4$. Therefore, Eq. (11.11) reduces to:

$$\begin{aligned} m_1 v_1 &= (m_1 + m_2)v_3 \\ v_1 &= \frac{(m_1 + m_2)v_3}{m_1} \end{aligned}$$

where m_1 is the mass of the bullet, m_2 is the mass of the pendulum, v_1 is the unknown velocity of the bullet, and v_3 is the post impact initial velocity as calculated in the first step. Before we proceed, all units have to be correct, so we must convert the units in the US system:

$$\begin{aligned} m_1 &= 1\,\text{oz}\left(\frac{1\,\text{lb}}{16\,\text{oz}}\right)\left(\frac{1\,\text{slug}}{32.2\,\text{lb}}\right) \\ &= 0.00194\,\text{slugs} \end{aligned}$$

$$\begin{aligned} m_2 &= 20\,\text{lb}\left(\frac{1\,\text{slug}}{32.2\,\text{lb}}\right) \\ &= 0.621\,\text{slugs} \end{aligned}$$

Now make the appropriate substitutions to solve the problem:

$$\begin{aligned} v_1 &= \frac{(m_1+m_2)v_3}{m_1} \\ &= \frac{(0.00194+0.621)(5.67)}{0.00194} \\ &= 1820\,\text{f/s} \end{aligned}$$

$$\begin{aligned} v_1 &= \frac{(m_1+m_2)v_3}{m_1} \\ &= \frac{(0.0283+9.07)(1.728)}{0.0283} \\ &= 555\,\text{m/s} \end{aligned}$$

Example 11.4. Apply the principle of conservation of linear momentum to a head-on collision between two cars. Both cars have the same weight, 4000 pounds (1814 kg). Assume the cars become entangled, so they do not rebound after the collision. Furthermore, one car is proceeding northbound at 60 mph (97 km/h) while the other is proceeding southbound at 40 mph (65 km/h) when they collide. Define north as positive and south as negative for the direction cosines. What are the vehicles' velocities after the collision?

Solution

First, we will calculate the momentum of each car. The first car has a momentum of 4000 pounds times 60 mph (1814 kg $\times$ 97 km/h) equaling 240,000 pound-miles per hour (175,958 kg-km/h). The second car has a momentum of 4000 pounds times 40 mph (1814 kg $\times$ 65 km/h) equaling 160,000

pound-miles per hour (117,910 kg-km/h). Since the first car is traveling north, the momentum vector is positive and the second car, traveling south, has a negative direction vector.

To obtain the total momentum of the system, we must add the two individual momenta together, keeping in mind the sign of each. This gives us:

US	SI
$240,000 - 160,000 = 80,000\,\text{lb-mph}$	$175,958 - 117,910 = 58,048\,\text{kg-kph}$

We will recall that the total momentum, $\mathbf{P_T}$, of a system before and after a collision is the same. Also, the resulting total momentum is positive, so the total momentum vector points north.

After the collision the total mass of this system of two cars has been concentrated into one mass of 8000 pounds (3628 kg), because the two cars stuck together. We should not be confused by our calling pounds a mass. We are still on the surface of the earth and gravity, g, appears on both sides of the equation and cancels out. Therefore, we can simply use the weights in pounds (the kilogram, kg, is already a mass unit). This mass will now have a post impact velocity expressed by the following equation:

US	SI
$(m_1+m_2)\delta v = \mathbf{P_T}$	$(m_1+m_2)\delta v = \mathbf{P_T}$
$\delta v = \dfrac{\mathbf{P_T}}{m_1+m_2}$	$\delta v = \dfrac{\mathbf{P_T}}{m_1+m_2}$
$\delta v = +\dfrac{80,000\,\text{lb}\cdot\text{mph}}{8000\,\text{lb}}$	$\delta v = +\dfrac{58,048\,\text{kg}\cdot\text{km/h}}{3628\,\text{kg}}$
$v = 10\,\text{mph}$	$v = 16\,\text{km/h}$

Since our direction cosine is positive, the post impact velocity is in the same direction as our northbound car. Therefore, the speed of the northbound car, which had been traveling 60 mph (97 km/h), has been reduced by 50 mph (81 km/h) down to 10 mph (16 km/h). The speed of the

southbound car has also been changed by 50 mph (81 km/h). Also, the southbound car has also undergone a change of direction. It was headed south at 40 mph (65 km/h) before the collision, but after the collision it was headed north at 10 mph (16 km/h).

We could probably think of other examples for which we could use the principle of conservation of linear momentum. As we progress further in our study of traffic crashes, we will learn methods that use this principle to reconstruct the speeds of vehicles involved in two-dimensional, or *planar* collisions. To do this it will be necessary for us to determine the speeds of the vehicles after impact as well as their directions before and after impact. With this information, we will then be able to find the speed of a vehicle at the time of impact, and will be able to relate this speed to the total speeds of the vehicles. We will examine these ideas more completely in Chapter 12.

11.4 Elastic and Inelastic Collisions

In our discussion of energy in Section 13.3 on page 13-10, we defined a conservative system as a system in which mechanical energy (kinetic energy + potential energy) is conserved. We also defined a nonconservative system as one in which mechanical energy is not conserved. If we apply these ideas to collisions, we can define elastic and inelastic collisions:[†]

> ELASTIC COLLISION
> A collision in which kinetic energy is conserved. It is theorized that the only true elastic collisions are those between sub-atomic particles.

> INELASTIC COLLISION
> A collision in which kinetic energy is not conserved. An example would be a collision of two lumps of clay that stick together after impact.

Real world collisions usually fall somewhere between those covered by these two definitions. A hard steel ball dropped on a hard steel plate will behave quite elastically. The steel ball will bounce quite a bit. A basketball dropped on a basketball court will tend more toward the elastic collision than toward the inelastic collision. However, the basketball will not have quite as much bounce as the steel ball.

[†]A collision occurs with little displacement over its duration. Hence, we need not consider a change of potential energy during the collision.

On the other hand, street and highway speed collisions between vehicles are almost totally inelastic. The vehicles crush significantly, and do not return the energy that has been expended in deforming the metal. The work done to crush the vehicles goes into other forms of energy, such as strain, heat and sound. However, some low-speed collisions do have some bounce.

We can model partially elastic collisions if we identify a *coefficient of restitution*. If two bodies collide in a collinear collision, then the coefficient of restitution is a dimensionless number given by:

$$\varepsilon = \frac{v_4 - v_3}{v_1 - v_2} \tag{11.15}$$

where v_3 and v_4 are post-impact velocities and v_1 and v_2 are pre-impact velocities.[‡] The coefficient of restitution can be between 0 and 1. From this definition we see that $v_1 - v_2$ is the relative velocity of approach, and $v_4 - v_3$ is the relative velocity of recession. If the second object is not moving (e.g., a floor or wall), then v_2 and v_4 are equal to zero.

We can use the concept of restitution to derive an in-line momentum equation that accounts for restitution. Let us start with Eq. (11.11), which is the conservation of linear momentum for an in-line collision. (Direction cosines are dropped for brevity.)

$$m_1 v_1 + m_2 v_2 = m_1 v_3 + m_2 v_4$$

Let us define the closing velocity as:

$$v_c = v_1 - v_2 \tag{11.16}$$

We can use the definition of restitution, Eq. (11.15), and the closing velocity, Eq. (11.16), to isolate v_4:

$$v_4 = \varepsilon v_c + v_3$$

Substitute this expression for v_4 into Eq. (11.11):

$$m_1 v_1 + m_2 v_2 = m_1 v_3 + m_2 [\varepsilon v_c + v_3]$$

[‡]The velocities used to determine the coefficient of restitution must be the components of the velocity vectors perpendicular to the impact plane.

Subtract $m_1 v_3$ and $m_2 v_2$ from both sides:

$$m_1 v_1 - m_1 v_3 = m_2[\varepsilon v_c + v_3] - m_2 v_2$$

Factor out an m_1 on the left-hand side and distribute m_2 on the right-hand side:

$$m_1(v_1 - v_3) = m_2 \varepsilon v_c + m_2 v_3 - m_2 v_2$$

Add $m_2 v_1$ to both sides:

$$m_1(v_1 - v_3) + m_2 v_1 = m_2 \varepsilon v_c + m_2 v_3 - m_2 v_2 + m_2 v_1$$

Subtract $m_2 v_3$ from both sides and factor out m_2:

$$m_1(v_1 - v_3) + m_2(v_1 - v_3) = m_2(\varepsilon v_c + v_1 - v_2)$$

Recall $v_1 - v_3 = \Delta v_1$ and $v_1 - v_2 = v_c$:

$$m_1 \Delta v_1 + m_2 \Delta v_1 = m_2(\varepsilon v_c + v_c)$$

Factor out Δv_1 on the left-hand side and v_c on the right-hand side:

$$\Delta v_1(m_1 + m_2) = m_2 v_c(\varepsilon + 1)$$

Solving for v_c yields:

$$\boxed{v_c = \frac{\Delta v_1(m_1 + m_2)}{m_2(\varepsilon + 1)}} \tag{11.17}$$

A similar derivation (except letting $v_3 = \varepsilon v_c - v_4$) will give closing velocity in terms of Δv_2:

$$\boxed{v_c = \frac{\Delta v_2(m_1 + m_2)}{m_1(\varepsilon + 1)}} \tag{11.18}$$

If the second object is sitting still, then the closing speed is v_1. Equations (11.17) and (11.18) are useful if the Δv of a vehicle is known from some other means, such as an event data recorder or a damage momentum analysis.

A perfectly elastic collision has a coefficient of restitution of 1, which means the relative velocities are the same both before and after the collision. An inelastic collision has a coefficient of restitution of 0, which means that there is no post collision relative velocities (the objects stick together). As the closing speed increases, the coefficient of restitution gets closer to zero. The values of ε have to be determined experimentally. Section **??** on page ?? continues the discussion of restitution.

Low speed traffic collisions may have a coefficient of restitution associated with the collision. If there is little or no permanent deformation, then some of the energy is returned to cause motion. Let us look at an example:

Example 11.5. A 3000-lb (1360-kg) car is stopped for a traffic light. It is rear-ended by a 5000-lb (2267-kg) pickup truck that is going 5 mph (8 km/h). All directions are positive in this case and the collision is bumper to bumper. If the coefficient of restitution is 0.3, what is the velocity of both the car and the pickup after the collision?

Begin with the formula from conservation of linear momentum, Eq. (11.11):

$$\underbrace{\delta_1 m_1 v_1}_{pickup} + \underbrace{\delta_2 m_2 v_2}_{car} = \underbrace{\delta_3 m_1 v_3}_{pickup} + \underbrace{\delta_4 m_2 v_4}_{car}$$

Because the car was stopped at impact, $v_2 = 0$. Also all direction cosines are $+1$ thus, the equation becomes:

$$m_1 v_1 = m_1 v_3 + m_2 v_4$$

Divide both sides by m_1:

$$v_1 = v_3 + \frac{m_2}{m_1} v_4 \tag{11.19}$$

From the formula for coefficient of restitution, Eq. (11.15):

$$\begin{aligned} \varepsilon &= \frac{v_4 - v_3}{v_1 - v_2} \\ 0.3 &= \frac{v_4 - v_3}{v_1} \\ 0.3 v_1 &= v_4 - v_3 \\ v_1 &= \frac{v_4 - v_3}{0.3} \end{aligned}$$

Substitute this expression for v_1 into Eq. (11.19):

$$\frac{v_4 - v_3}{0.3} = v_3 + \frac{m_2}{m_1} v_4$$

Group like terms and solve for v_3:

$$v_4 - v_3 = 0.3v_3 + 0.3\frac{m_2}{m_1} v_4$$

Move $0.3v_3$ to the left side by subtracting:

$$v_4 - 1.3v_3 = 0.3\frac{m_2}{m_1} v_4$$

Move the v_4 to the right side by subtracting:

$$-1.3v_3 = 0.3\frac{m_2}{m_1} v_4 - v_4$$

Multiply through by -1 to get rid of the minus signs:

$$1.3v_3 = v_4 - 0.3\frac{m_2}{m_1} v_4$$

Factor out v_4 on the right side:

$$1.3v_3 = \left(1 - 0.3\frac{m_2}{m_1}\right) v_4$$

Divide through by 1.3 to isolate v_3:

$$v_3 = \left(\frac{1}{1.3} - \frac{0.3m_2}{1.3m_1}\right) v_4$$

$$v_3 = \left(0.77 - 0.23\frac{m_2}{m_1}\right) v_4$$

Since the ratio of the masses are the same regardless of units of measure, we can compute the ratio:

$$\frac{m_2}{m_1} = \frac{3000}{5000} = \frac{1360}{2267} = 0.60$$

Now we can finish solving for v_3 in terms of v_4:

$$v_3 = (0.77 - 0.23(0.60))v_4$$
$$v_3 = 0.632v_4 \tag{11.20}$$

Note that the expression here is an expression for v_3 in terms of v_4. We still have to calculate v_3. Substitute the expression for v_3 from Eq. (11.20) into Eq. (11.19) and solve for v_4:

$$\begin{aligned} v_1 &= v_3 + \frac{m_2}{m_1}v_4 \\ &= 0.632v_4 + \frac{m_2}{m_1}v_4 \\ &= 0.632v_4 + 0.60v_4 \\ v_1 &= 1.232v_4 \\ v_4 &= \frac{v_1}{1.232} \end{aligned}$$

US	SI
$v_4 = \frac{5}{1.232}$	$v_4 = \frac{8}{1.232}$
$v_4 = 4.06\,\text{mph}$	$v_4 = 6.49\,\text{kph}$

Return to Eq. (11.19) and solve for v_3:

$$v_3 = v_1 - \frac{m_2}{m_1}v_4$$

US	SI
$v_3 = 5 - 0.60(4.06)$	$v_3 = 8 - 0.60(6.49)$
$v_3 = 2.56\,\text{mph}$	$v_3 = 4.10\,\text{kph}$

We can check our results with the momentum equation:

US	SI
$m_1 v_1 = m_1 v_3 + m_2 v_4$	$m_1 v_1 = m_1 v_3 + m_2 v_4$
$5000(5) = 5000(2.56) + 3000(4.06)$	$2267(8) = 2267(4.10) + 1360(6.50)$
$25,000 = 12,800 + 12,180$	$18,136 = 9,294.70 + 8,840$
$25,000 \approx 24,980\,\text{lb}\cdot\text{mph}$	$18,136 \approx 18,134\,\text{kg}\cdot\text{kph}$

These numbers are virtually identical![§] Thus, the post-impact speed of the car is 4.06 mph (6.49 km/h), and the post-impact speed of the truck is 2.56 mph (4.1 km/h).

11.5 The Presence of External Forces

In some situations, such as low-speed impacts, sideswipes, and striking large immovable objects, the ever-present external forces can achieve the same significance as the collision forces and must not be excluded from the analysis. Reconsider Figure 11.2 with a significant external force resisting

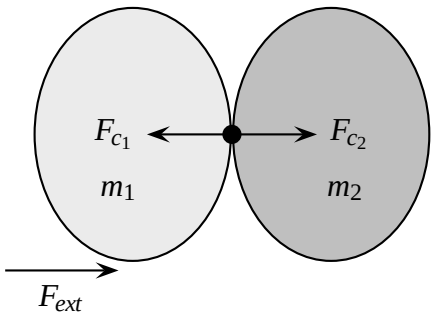


Figure 11.3: A collision between two objects with a significant external force present during the collision.

[§] Carrying one more decimal point would have given an equality.

the movement of object #1, as shown in Figure 11.3.

We can separate the two bodies in Figure 11.3, sum all the forces, and use Newton's Second Law to write:

$$\begin{aligned} m_1 a_1 + F_{ext} &= F_{c_1} \\ m_2 a_2 &= F_{c_2} \end{aligned}$$

The kinematic definition of a is:

$$a = \frac{\Delta v}{\Delta t} \tag{11.21}$$

which means for each object:

$$a_1 = \frac{v_1 - v_3}{t_1 - t_3} \qquad \text{and} \qquad a_2 = \frac{v_2 - v_4}{t_2 - t_4}$$

If subscript $_1$ refers to the status of the first object before the collision and subscript $_3$ refers to the status after the collision, then $t_3 > t_1$. Similarly, $t_4 > t_2$, so the denominator of Eq. (11.21) is negative given the definition of $\Delta v = v_o - v_f$. Thus:

$$a = -\frac{\Delta v}{\Delta t}$$

Newton's Third Law states that the collision forces must be equal and opposite:

$$\begin{aligned} F_{c_1} &= -F_{c_2} \\ m_1 a_1 + F_{ext} &= -m_2 a_2 \\ m_1 \left(-\frac{\Delta v_1}{\Delta t} \right) + F_{ext} &= -m_2 \left(-\frac{\Delta v_2}{\Delta t} \right) \end{aligned}$$

Multiply through by Δt:

$$\begin{aligned} -m_1\Delta v_1 + F_{ext}\Delta t &= m_2\Delta v_2 \\ -m_1(v_1 - v_3) + F_{ext}\Delta t &= m_2(v_2 - v_4) \\ -m_1v_1 + m_1v_3 + F_{ext}\Delta t &= m_2v_2 - m_2v_4 \end{aligned}$$

Rearrange to get the impulse-momentum relationship:

$$\boxed{m_1v_1 + m_2v_2 = m_1v_3 + m_2v_4 + F_{ext}\Delta t} \tag{11.22}$$

which says that the incoming momentum must equal the outgoing momentum plus what is "lost" due to an external impulse. It is important to note that the sign of the external force was such that it resisted the acceleration of mass #1.

If a vehicle strikes a tree, is momentum still conserved? No, the large external force imparted by the tree countered all the momentum of the vehicle. Newton's Laws of Motion are still satisfied but the conservation of momentum does not hold, this is why COLM is a *principle* rather than a law of motion.

Example 11.6. When we solved Example 11.5, we made the assumption that the collision forces were large compared to the ground forces acting on the vehicles. In low speed collisions, this may not be the case. Let us reexamine our example problem with this in mind. The solution that we calculated in Example 11.5 would be valid if the brakes on the car were off. Let us look at what would happen if the brakes were locked and the coefficient of friction, $\mu = 0.75$.

Solution

We begin the analysis by calculating the ground frictional force of vehicle #2 from Example 11.5, which now has its brakes locked and slides on a surface with a coefficient of friction of $\mu = 0.75$.

We use the following equation:

$$F = \mu w$$

US	SI
$F = \mu w$	$F = \mu mg$
$F = (0.75)3000$	$F = (0.75)(1360)(9.81)$
$F = 2250\,\text{lb}$	$F = 10,006\,\text{N}$

Why have we made this force calculation? Since the brakes on the struck vehicle are locked, then there will be the additional force from the friction of the tires on the road resisting the collision force that is trying to accelerate the vehicle. Remember, it is the presence of external, unbalanced forces that causes acceleration.

The duration of time over which this frictional force acts is the collision time because this is a reaction force. For this case, we will consider the collision time to be 0.100 seconds (100 milliseconds). The product of force and the change in time is impulse. Thus, impulse will have to appear in our COLM equation, as it is consumed by an additional force that we must account for.

$$\underbrace{\delta_1 m_1 v_1}_{pickup} + \underbrace{\delta_2 m_2 v_2}_{car} = \underbrace{\delta_3 m_1 v_3}_{pickup} + \underbrace{\delta_4 m_2 v_4}_{car} + \underbrace{F_g \Delta t}_{impulse} \tag{11.23}$$

where $F_g = F_{ext}$, the ground frictional force of vehicle #2. Let's assume all direction cosines are positive as they were in the previous example. We can take Eq. (11.23) and solve for v_1 (with $v_2 = 0$ since it was stopped):

$$v_1 = v_3 + \frac{m_2}{m_1} v_4 + \frac{F_g \Delta t}{m_1} \tag{11.24}$$

From the coefficient of restitution equation:

$$v_1 = \frac{v_4 - v_3}{0.3}$$

Substitute into the v_1 equation:

$$\frac{v_4 - v_3}{0.3} = v_3 + \frac{m_2}{m_1} v_4 + \frac{F_g \Delta t}{m_1}$$

Recall that the mass ratio $\frac{m_2}{m_1}$ for this problem is 0.60.

$$\frac{v_4 - v_3}{0.3} = v_3 + 0.60v_4 + \frac{F_g \Delta t}{m_1}$$

Rearrange, solving for v_3 in terms of v_4:

$$\begin{aligned} v_4 - v_3 &= 0.3v_3 + 0.18v_4 + 0.3\left(\frac{F_g \Delta t}{m_1}\right) \\ 1.3v_3 &= v_4 - 0.18v_4 - 0.3\left(\frac{F_g \Delta t}{m_1}\right) \\ v_3 &= \frac{0.82}{1.3}v_4 - \frac{0.3}{1.3}\left(\frac{F_g \Delta t}{m_1}\right) \\ v_3 &= 0.63v_4 - 0.23\left(\frac{F_g \Delta t}{m_1}\right) \end{aligned}$$

US

Recall that $m = w/g$:

$$\begin{aligned} v_3 &= 0.63v_4 - 0.23\left(\frac{gF_g \Delta t}{w_1}\right) \\ v_3 &= 0.63v_4 - 0.23\left(\frac{(32.2)F_g \Delta t}{w_1}\right) \end{aligned}$$

Substitute values for F_g, Δt and w_1:

$$\begin{aligned} v_3 &= 0.63v_4 - 0.23\left(\frac{(32.2)F_g \Delta t}{w_1}\right) \\ v_3 &= 0.63v_4 - 0.23\left(\frac{(32.2)(2250)(0.100)}{5000}\right) \\ v_3 &= 0.63v_4 - 0.334 \end{aligned}$$

SI

Recall that kilograms is a mass unit.

Substitute values for F_g, Δt and m_1:

$$\begin{aligned} v_3 &= 0.63v_4 - 0.23\left(\frac{F_g \Delta t}{m_1}\right) \\ v_3 &= 0.63v_4 - 0.23\left(\frac{10,006(0.100)}{2273}\right) \\ v_3 &= 0.63v_4 - 0.101 \end{aligned}$$

US	SI

Substitute this value for v_3 into the v_1 equation while recalling the mass ratio is 0.60:

$$v_1 = v_3 + \frac{m_2}{m_1}v_4 + \frac{F_g\Delta t}{m_1}$$
$$v_1 = (0.63v_4 - 0.334) + 0.60v_4 + \frac{225}{\frac{5000}{32.2}}$$
$$v_1 = (0.63v_4 - 0.334) + 0.60v_4 + 1.449$$
$$v_1 = 1.23v_4 + 1.115$$

Now solve for v_4, as we already know $v_1 = 5\,\text{mph} = 7.33\,\text{fps}$.

$$1.23v_4 = v_1 - 1.115$$
$$v_4 = \frac{v_1}{1.23} - 0.906$$
$$v_4 = \frac{7.33}{1.23} - 0.906$$
$$v_4 = 5.05\,\text{ft/sec}$$

$$v_1 = v_3 + \frac{m_2}{m_1}v_4 + \frac{F_g\Delta t}{m_1}$$
$$v_1 = (0.63v_4 - 0.101) + 0.60v_4 + \frac{1000.6}{2267}$$
$$v_1 = (0.63v_4 - 0.101) + 0.60v_4 + 0.441$$
$$v_1 = 1.23v_4 + 0.340$$

Now solve for v_4, as we already know $v_1 = 8\,\text{kph} = 2.22\,\text{m/s}$.

$$1.23v_4 = v_1 - 0.340$$
$$v_4 = \frac{v_1}{1.23} - 0.276$$
$$v_4 = \frac{2.22}{1.23} - 0.276$$
$$v_4 = 1.52\,\text{m/sec}$$

Now solve for v_3 using Eq. (11.24) written in terms of v_3:

$$v_3 = v_1 - \frac{m_2}{m_1}v_4 - \frac{F_g\Delta t}{m_1}$$

US	SI
$v_3 = v_1 - 0.60v_4 - 1.449$	$v_3 = v_1 - 0.60v_4 - 0.441$
$v_3 = 7.33 - 0.60(5.05) - 1.449$	$v_3 = 2.22 - 0.60(1.52) - 0.441$
$v_3 = 2.85\,\text{fps}$	$v_3 = 0.867\,\text{m/s}$

Check with conservation of linear momentum, Eq. (11.24):

$$m_1v_1 + m_2v_2 = m_1v_3 + m_2v_4 + F_g\Delta t$$

$\frac{5000(7.33)}{32.2} = \frac{5000(2.85)}{32.2} + \frac{3000(5.05)}{32.2} + 225$	$2267(2.22) = 2267(0.867) + 1360(1.52) + 1000.6$
$1138 = 1138$	$5032 \approx 5033$

The momentum equation checks our results! By the way, this is an example of solving simultaneous equations by the substitution method.

In this example, we see that the post-impact velocity of the car without brakes being locked is about 4 mph (6.5 km/h), whereas its post-impact velocity with brakes being applied is about 3.4 mph (5.59 km/h). Thus, we must consider ground forces in low speed vehicle collisions. If we are looking at occupant loads (forces) in low speed collisions, these differences can be important.

Example 11.7. Consider a car cruising down the road and, after suddenly slamming on the brakes, comes to a stop. Is momentum conserved in this maneuver?

This is a case in which there are external forces present. We cannot draw a box around our system and safely ignore the presence of external forces. If we consider the braking maneuver as an impulse, then we can say

$$m\Delta v = F\Delta t$$

We'll assume a level surface and all wheels locked, so force due to friction is $F = \mu w$. Also recall that $m = \frac{w}{g}$, so:

$$\frac{w}{g}\Delta v = \mu w \Delta t$$

Cancel weight from both sides and divide through by Δt:

$$\frac{\Delta v}{g\Delta t} = \mu$$

Now recall the definition of average acceleration, $a = \frac{\Delta v}{\Delta t}$, so

$$\frac{a}{g} = \mu$$

This demonstrates that the coefficient of friction is also a ratio of accelerations. This concept is explored in more detail in Chapter 9. To answer the original question, momentum is not conserved according to Eq. (11.7), but the concept of impulse and momentum obtained from Newton's Laws still holds true.

Example 11.8. A 3000-lb (1360-kg) vehicle wraps around a tree at 30 mph (48 kph). The vehicle has 90,000 lb-mph (65,280 kg-kph) of momentum coming into the collision and no momentum coming out. How is momentum conserved?

Again, Eq. (11.7) does not apply here because there is a very significant impulse during the collision that matches the change in momentum. If a time of collision is around 0.100 seconds, then the time average of the collision force would be:

US

$$\begin{aligned} F &= \frac{m\Delta v}{\Delta t} \\ &= \frac{w\Delta v}{g\Delta t} \\ &= \frac{90,000}{32.2(0.100)}\left(\frac{1.466\,\text{fps}}{1\,\text{mph}}\right) \\ &= 40,975\,\text{lb} \end{aligned}$$

SI

$$\begin{aligned} F &= \frac{m\Delta v}{\Delta t} \\ &= \frac{65,280}{0.100}\left(\frac{1\,\text{m/s}}{3.6\,\text{kph}}\right) \\ &= 181,333\,\text{N} \end{aligned}$$

Exercises

Exercise 11.1. Show that the dimensions for impulse (lb-sec) are equivalent to the dimensions for momentum (slug-ft/sec).

Exercise 11.2. Show that the dimensions for impulse (N-sec) are equivalent to the dimensions for momentum (kg-m/sec).

Exercise 11.3. Develop a conversion factor to convert units of momentum between lb-mph and slug-ft/s.

$\boxed{0.0455}$

Exercise 11.4. Develop a conversion factor to convert units of momentum between kg-kph and kg-m/s.

$\boxed{1/3.6}$

Exercise 11.5. A 3500-lb car is waiting to turn left at an intersection and a 4000 lb SUV strikes the car in the right door in a "T-bone" configuration. Both cars move off together with a speed of 25 mph. How fast was the SUV going when it hit the car?

$\boxed{46.87\,\text{mph}}$

Exercise 11.6. A 3200-kg van is cruising down the road at 80 kph. A 1500 kg car, heading in the same direction as the van, rear ends the van and both vehicles leave the collision at a speed of 95 kph. What is the impact speed of the car?

$\boxed{127\,\text{kph}}$

Exercise 11.7. What speed would a 4800-lb pickup have to be going in order to stop a 48,000-lb dump truck traveling at 10 mph after they collide head-on?

$\boxed{100\,\text{mph}}$

Exercise 11.8. During a frontal crash test, an 1800-kg vehicle is run into a barrier at 56 kph and rebounds at 5 kph.

a.) Determine Δv. $\boxed{61\,\text{kph}}$

b.) Determine the coefficient of restitution $\boxed{0.089}$

c.) Calculate the change in momentum. $\boxed{91,800\,\text{kg-kph}}$

d.) If the impact duration was 120 ms, what was the average force exerted on the vehicle? $\boxed{212,500\,\text{N}}$

Exercise 11.9. A 3800-pound vehicle is stationary with its brakes locked on a level surface that has a friction coefficient of 0.75. A constant external force of 8500 pounds acts on the vehicle for 0.100 seconds.

a.) How fast is the vehicle going at the end of the force application? $\boxed{4.78\,\text{ft/s}}$

b.) If the brakes remain locked, how far will it travel after the force ceases? $\boxed{0.473\,\text{ft}}$

c.) If the brakes are not locked, and the rolling coefficient is 0.02, how fast will the vehicle be going at the end of the force application? $\boxed{7.13\,\text{ft/s}}$

d.) How far will the vehicle in question c.) roll after the force ceases to act? $\boxed{39.71\,\text{ft}}$

e.) How fast will the vehicle be going if all brakes are locked and the uniform force is 40,000 pounds rather than 8500 pounds? $\boxed{31.47\,\text{ft/s}}$

f.) How fast will the vehicle in part e.) be going if the brakes are not locked? $\boxed{33.83\,\text{ft/s}}$

Exercise 11.10. A 3200-kg van is cruising down the road at 80 kph. A 1500 kg car, heading in the opposite direction of the van, hits the van head-on. The two vehicles become entangled and the van pushes the car back. The post collision speed is 15 kph. What is the impact speed of the car? $\boxed{123.6\,\text{kph}}$

Exercise 11.11. A 130-pound person is reported to have fallen 145 feet from the 17th floor of a building. The person landed on a metal ventilator box, which this person crushed to a depth of 18 inches. Neglecting air resistance, calculate:

a.) The speed of the person just before impact with the box. $\boxed{96.6\,\text{ft/s}}$

b.) The average acceleration of the person while in contact with the box. $\boxed{-3112\,\text{ft/s}^2}$

c.) The time it took to crush the box. $\boxed{0.031\,\text{sec}}$

d.) The person's average force while crushing the box. $\boxed{12,564\,\text{lb}}$

e.) The person's potential energy at the start of the fall. $\boxed{18,850\,\text{ft-lb}}$

f.) The person's potential energy just before the impact with the box. $\boxed{0\,\text{ft-lb}}$

g.) The work done by the person in crushing the box. $\boxed{18,850\,\text{ft-lb}}$

h.) The person's impulse while crushing the box. $\boxed{389.4\,\text{lb-sec}}$

i.) The person's change in momentum while crushing the box. $\boxed{390\,\text{lb-sec}}$

12 Collision Analysis Using Conservation of Linear Momentum

12.1 Introduction

The purpose of this chapter is to prepare the investigator to use Conservation of Linear Momentum (COLM) for collision analysis. In Chapter 11, we developed the basic principles and equations for this analysis. An understanding of these concepts is prerequisite to reading this chapter. Conservation of Linear Momentum, when applied properly, is one of the most powerful tools in the reconstructionist's toolbox for the determination of collision impact speeds.

The first section of this chapter defines the types of collisions and configurations that we can analyze using COLM. The second section of the chapter presents examples of analyzing collinear collisions and builds on the concepts presented in Chapter 11. The third section of this chapter expands into planar problems in which the vehicles are coming in from different directions. The two dimensional momentum (planar) problems will be solved both graphically and mathematically. The fourth section discusses the evidence required to solve the COLM problem. The chapter concludes with a discussion of special considerations when analyzing collisions using conservation of linear momentum.

For the analysis presented in this chapter, we will assume any external forces are negligible when compared to the collision forces. Additionally, we will define all of our collisions as *inelastic*, which means that mechanical energy (the sum of potential energy and kinetic energy) is *not* conserved. However, since we consider potential energy changes to be negligible during the collision phase of the crash, we may simplify this definition to address kinetic energy only.

12.2 Collision Types and Configurations

In this section, our discussion is in reference to the momentum and force vectors associated with the collision. The momentum vectors are collinear with the velocity vectors regardless of the heading (where the headlights are pointing) of the vehicle. There is no requirement that the heading of the vehicle be collinear with the velocity vector. The direction in which the force (impulse) vectors approach the vehicles with respect to their velocity vectors is called the *Principal Direction of Force* (PDOF). The PDOF is the vector sum of all of the forces acting on the vehicles through the collision sequence. Because of Newton's Third Law, the PDOF for one vehicle must be equal and opposite to the PDOF of the other vehicle, regardless of vehicle size.

12.2.1 Collinear, Central Collisions

A collinear, central collision is a configuration in which the velocity vectors of the colliding vehicles are parallel and the PDOF goes through the center of mass of each vehicle. A collinear collision is also sometimes referred to as an *in-line* collision. If one vehicle or movable object is stopped, then the pre-impact velocity vector of it will be zero. The two objects in the collision may not both be vehicles. For example, a car may run into a movable dumpster and both will move to some final rest position.

In a collinear, central, inelastic collision, we expect the two vehicles to move off essentially as one unit. Therefore, the magnitude of the velocity for each vehicle is the same after impact and the two units may be treated as a single mass. There will be no rotation in either vehicle induced from collision forces, but vehicles may separate and have minor rotation after the impact as the vehicles move to final rest.

12.2.2 Collinear, Non-Central Collisions

A collinear, non-central collision is a collision in which the velocity vectors of the colliding vehicles are parallel, but the PDOF does not go through the center of mass. Non-central collisions are sometimes called *offset* collisions. Again, if one vehicle or movable object is stopped, its velocity vector is zero.

We do not expect both vehicles to have the same magnitude of velocity after a collinear, non-central, inelastic collision. We also expect rotation on the part of at least one of the vehicles due

to the collision forces involved. We consider both vehicles to have parallel post-impact velocity vectors immediately after impact.

12.2.3 Two-Dimensional, Central Collisions

Two-dimensional collisions are those collisions in which the approach momentum (velocity) vectors of each vehicle are not collinear. An intersection collision in which one vehicle runs a red light and collides with another at some angle is an example of this configuration. In this collision type, "central" refers to the PDOF passing through the center of mass of each vehicle.

With central collisions, we would expect the vehicles to move off as one unit with one velocity vector. We also expect no rotation of either vehicle because of the collision forces. The vehicles may separate later, during the post-impact trajectory.

Some reconstructionists refer to two-dimensional collisions as "angular collisions" and then further say that the analysis is "angular momentum." Although this may be convenient, it is not physically correct. Angular momentum is defined in Chapter 14.

12.2.4 Two-Dimensional, Non-Central Collisions

The two-dimensional, non-central collision is a configuration in which the approach velocity vectors are not collinear and the PDOF of at least one of the vehicles does not go through the center of mass. In this inelastic collision, we expect to see rotation on the part of the vehicle with the non-central PDOF as a result of the collision forces. Additionally, there is no physical requirement that the post-impact velocity of each vehicle be equal in either magnitude or direction.

12.3 Collision Analysis Examples in One Dimension

12.3.1 A Moving Vehicle into a Stopped Vehicle

Example 12.1. A traffic jam develops and vehicle #1 does not stop and rear-ends vehicle #2. Witnesses say vehicle #2 was stopped. The post-impact speeds were determined to be 12 mph (19.3 kph) for vehicle #1 and 15 mph (24.1 kph) for vehicle #2. How fast was vehicle #1 going upon impact if vehicle #1 weighs 3400 lb (1542 kg) and vehicle #2 weighs 4300 lb (1950 kg)? Both

vehicles end up down road from the area of impact. This is an example of a collinear, non-central (offset) collision.

Solution

The direction cosines in this case are all positive. The pre-impact speed of vehicle #2 is zero so Eq. (11.12) on page 11-11 becomes:

$$w_1 v_1 = w_1 v_3 + w_2 v_4$$

Solve for v_1:

$$v_1 = v_3 + \frac{w_2}{w_1} v_4$$

US	SI
$v_1 = 12 + \left(\frac{4300}{3400}\right) 15$	$v_1 = 19.3 + \left(\frac{1950}{1542}\right) 24.1$
$= 30.97\,\text{mph}$	$= 49.77\,\text{kph}$

What is the coefficient of restitution in this crash? From the definition of the coefficient of restitution, we have

$$\varepsilon = \frac{v_4 - v_3}{v_1 - v_2}$$

US	SI
$\varepsilon = \dfrac{15-12}{30.97-0}$	$\varepsilon = \dfrac{24.1-19.3}{49.77-0}$
$= 0.096$	$= 0.096$

If the exit speed of one of the vehicles is not known, the problem can be solved by estimating a coefficient of restitution and making the appropriate substitutions. Furthermore, assuming no restitution tends to make the estimation of the closing speed $(v_1 - v_2)$ higher. Restitution decreases as approach speeds increase.

12.3.2 Vehicles Colliding in the Same Direction

Example 12.2. Vehicle #1 weighs 3220 lb (1464 kg) and vehicle #2 weighs 4340 lb (1973 kg). Vehicle #2 is going 25 mph (40 kph) when it is rear-ended by vehicle #1. Vehicle #1 travels 75 ft (23 m) post-impact with a drag factor of 0.59. Vehicle #2 travels 110 ft (33.5 m) with a drag factor of 0.38.

a.) What is the impact speed of vehicle #1?

b.) What is the Δv of each vehicle?

Solution

a.) Begin with Eq. (11.12), which is the conservation of linear momentum including direction cosines:

$$\delta_1 w_1 v_1 + \delta_2 w_2 v_2 = \delta_3 w_1 v_3 + \delta_4 w_2 v_4$$

Write down the direction cosines:

$$\delta_1 = +1, \quad \delta_2 = +1, \quad \delta_3 = +1, \quad \delta_4 = +1$$

Make the appropriate substitutions:

$$w_1 v_1 + w_2 v_2 = w_1 v_3 + w_2 v_4$$

Now solve for v_1:

$$w_1 v_1 = w_1 v_3 + w_2 v_4 - w_2 v_2$$

$$v_1 = \frac{w_1 v_3 + w_2 v_4 - w_2 v_2}{w_1}$$

$$v_1 = v_3 + \frac{w_2}{w_1} v_4 - \frac{w_2}{w_1} v_2$$

We were given the following information:

US	SI
$w_1 = 3220\,\text{lb}$	$w_1 = 1464\,\text{kg}$
$w_2 = 4340\,\text{lb}$	$w_2 = 1973\,\text{kg}$
$v_2 = 25\,\text{mph}$	$v_2 = 40\,\text{kph}$
$d_3 = 75\,\text{ft}$	$d_3 = 23\,\text{m}$
$f_3 = 0.59$	$f_3 = 0.59$
$d_4 = 110\,\text{ft}$	$d_4 = 33.5\,\text{m}$
$f_4 = 0.38$	$f_4 = 0.38$

We need to calculate v_3 and v_4 based on the slide-to-stop equation:

US	SI
$v_3 = \sqrt{30 d_3 f_3}$	$v_3 = \sqrt{254 d_3 f_3}$
$= \sqrt{30(75)(0.59)}$	$= \sqrt{254(23)(0.59)}$
$= 36.43\,\text{mph}$	$= 58.70\,\text{kph}$
$v_4 = \sqrt{30 d_4 f_4}$	$v_4 = \sqrt{254 d_4 f_4}$
$= \sqrt{30(110)(0.38)}$	$= \sqrt{254(33.5)(0.38)}$
$= 35.41\,\text{mph}$	$= 56.86\,\text{mph}$

Now we have all the information needed to solve for v_1:

US	SI
$v_1 = v_3 + \frac{w_2}{w_1}v_4 - \frac{w_2}{w_1}v_2$	$v_1 = v_3 + \frac{w_2}{w_1}v_4 - \frac{w_2}{w_1}v_2$
$= 36.43 + \frac{4340}{3220}(35.41) - \frac{4340}{3220}(25)$	$= 58.70 + \frac{1973}{1464}(56.86) - \frac{1973}{1464}(40)$
$= 50.46\,\text{mph}$	$= 81.42\,\text{kph}$

Thus, vehicle #1 was traveling 50.46 mph (81.42 kph) at impact.

b.) Now calculate Δv for each vehicle. Since the velocity vectors and Δv vectors are collinear and pointed in the same direction, the result of vector addition is the same as scalar addition. We will use the definition of Δv that says:

$$\Delta v = v_o - v_f$$

Notice that a positive Δv corresponds to a vehicle slowing down. For vehicle #1:

US	SI
$\Delta v_1 = v_1 - v_3$	$\Delta v_1 = v_1 - v_3$
$= 50.46 - 36.43$	$= 81.42 - 58.70$
$= 14.03\,\text{mph}$	$= 22.72\,\text{kph}$

This means vehicle #1 was slowed during the collision. Similarly, for vehicle #2:

US	SI
$\Delta v_2 = v_2 - v_4$	$\Delta v_2 = v_2 - v_4$
$= 25 - 35.41$	$= 40 - 56.86$
$= -10.41\,\text{mph}$	$= -16.86\,\text{kph}$

By our definition of Δv, vehicle #2 was accelerated by the impact.

As a check on our work, we can verify our results using Newton's Third Law, which states that the collision impulses must be equal and opposite. Since impulse is equal to a change in momentum, we can write:

$$w_1 \Delta v_1 = -w_2 \Delta v_2$$

A valid solution for the change in velocities must satisfy Newton's Third Law. By rearranging and solving for Δv_2 we get:

US	SI
$\Delta v_2 = -\frac{w_1}{w_2}\Delta v_1$	$\Delta v_2 = -\frac{w_1}{w_2}\Delta v_1$
$= -\frac{3220}{4340}(14.03)$	$= -\frac{1464}{1973}(22.72)$
$= -10.41\,\text{mph}$ check!	$= -16.86\,\text{kph}$ check!

12.3.3 Vehicles Colliding in Opposite Directions

Example 12.3. Vehicle #1 weighs 3220 lb (1464 kg) and vehicle #2 weighs 4340 lb (1973 kg). Vehicle #2 is going 25 mph (40 kph) when it is hit offset by vehicle #1. There is enough offset to spin the vehicles 180° and vehicle #1 exits the impact with a speed of 10 mph (16 kph) in the same direction it came in from. Similarly, vehicle #2 had a post-impact velocity of 10 mph (16 kph) in the same direction it came in from. What is the impact speed of vehicle #1?

Solution

Begin with Eq. (11.12), which is the conservation of linear momentum including direction cosines:

$$\delta_1 w_1 v_1 + \delta_2 w_2 v_2 = \delta_3 w_1 v_3 + \delta_4 w_2 v_4$$

Writing down the direction cosines requires a little more thought. If we define the initial direction of travel of vehicle #1 as positive, then the direction cosines are:

$$\delta_1 = +1, \quad \delta_2 = -1, \quad \delta_3 = +1, \quad \delta_4 = -1$$

Make the appropriate substitutions:

$$w_1 v_1 - w_2 v_2 = w_1 v_3 - w_2 v_4$$

Now solve for v_1:

$$w_1 v_1 = w_1 v_3 - w_2 v_4 + w_2 v_2$$

$$v_1 = \frac{w_1 v_3 - w_2 v_4 + w_2 v_2}{w_1}$$

$$v_1 = v_3 - \frac{w_2}{w_1} v_4 + \frac{w_2}{w_1} v_2$$

Input the given information to solve for impact speed of vehicle #1:

US	SI
$v_1 = v_3 - \frac{w_2}{w_1} v_4 + \frac{w_2}{w_1} v_2$	$v_1 = v_3 - \frac{w_2}{w_1} v_4 + \frac{w_2}{w_1} v_2$
$= 10 - \frac{4340}{3220}(10) + \frac{4340}{3220}(25)$	$= 16 - \frac{1973}{1464}(16) + \frac{1973}{1464}(40)$
$= 30.21\,\text{mph}$	$= 48.34\,\text{kph}$

12.4 Collision Analysis in Two Dimensions

Up to this point, we have discussed collisions in one dimension. However, many of the traffic collisions that we investigate are two-dimensional. The approach velocity vectors of the colliding vehicles are not collinear. After the collision, the vehicles may or may not stick together and the direction of the departure velocity vectors will be different from the approach. To solve two-

dimensional collisions we must remember that momentum is a vector because it is the product of a scalar quantity (mass) and a vector quantity (velocity). Thus, we will need to take into account both the magnitude and directional component of the approach and departure momentum vectors. We can solve two-dimensional collisions graphically and mathematically. Let's first look at the general concepts of a graphical analysis of a two-dimensional collision and consider the information that can be obtained. We will then examine the mathematical analysis and the information that can be obtained.

12.4.1 Graphical Analysis

A graphical solution to a two-dimensional collision involves creating a *vector diagram*. This involves plotting momentum vectors on a coordinate system and using it for analysis. Recalling from Chapter 11, the principal of COLM states that the total pre-impact momentum equals the total post-impact momentum. The total pre-impact momentum consists of the momenta brought to the collision by each vehicle. The total post-impact momentum consists of the momenta taken from the collision by each vehicle. Therefore, if the total post-collision momentum can be graphically determined, then the total pre-collision momentum will be known. Knowing the total pre-collision momentum will, in turn, allow the impact speeds of the two vehicles to be calculated.

An overview of the steps of performing a graphical solution is as follows:

a.) Assign one vehicle as vehicle #1. Its approach momentum vector will be on the positive *x*-axis (zero-degree approach angle).

b.) Plot the departure momentum vectors for each vehicle, to scale, on a coordinate system.

c.) Add, in a vector sense, the two departure momentum vectors to obtain the total post-collision momentum vector.

d.) The total post-collision momentum vector also represents the total pre-collision momentum vector.

e.) Resolve the total pre-collision momentum vector into its components. Each component is the approach momentum vector for the respective vehicle.

f.) Obtain the impact speeds of each vehicle from its pre-collision momentum vector based on the scale of the drawing.

We begin by establishing a coordinate system to use.

12.4.1.1 Coordinate System

To graphically solve a two-dimensional collision, we use polar coordinates. Figure 12.1 shows the components of a polar coordinate system. The layout of the axes in a polar coordinate system is similar to that of a rectangular, or Cartesian, coordinate system. Refer to Section **??** on page ?? for further details.

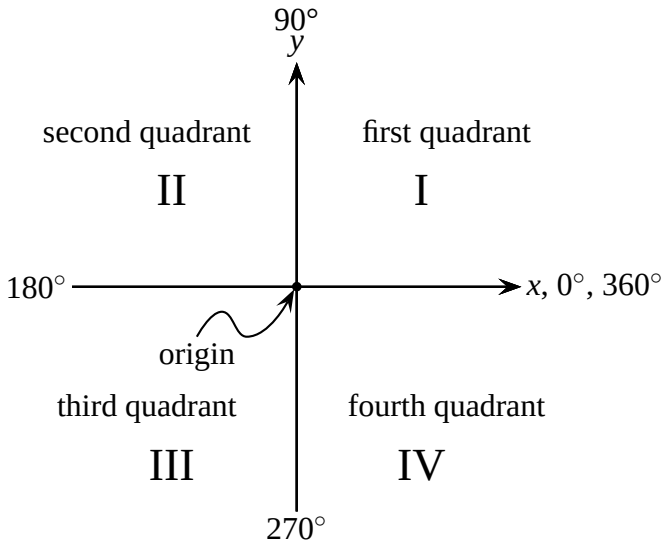


Figure 12.1: The polar coordinate system used in collision analysis using conservation of linear momentum.

The *x*- and *y*- axes are at right angles to each other. Approach and departure momentum vectors are drawn with their tails located at the origin, *O*. Angles for approach and departure momentum vectors are measured in degrees, counterclockwise with respect to the *x*-axis.

12.4.1.2 Vector Addition

If we wish to add two vectors together to obtain a *resultant* vector, we can perform this vector addition several different ways. One way is called the "head-to-tail" method, another is called the "parallelogram method." Another way is to resolve each of the two vectors into their *x*-direction and *y*-direction vector components, as in Figure 12.2. A discussion of these methods of vector addition can be found in Chapter 6.

Once the *x*- and *y*-components of each vector are found, we add the magnitudes of the *x*-direction vector components together and add the magnitudes of the *y*-direction vector components together. This gives us a total *x*-direction vector and a total *y*-direction vector. These vectors become, in

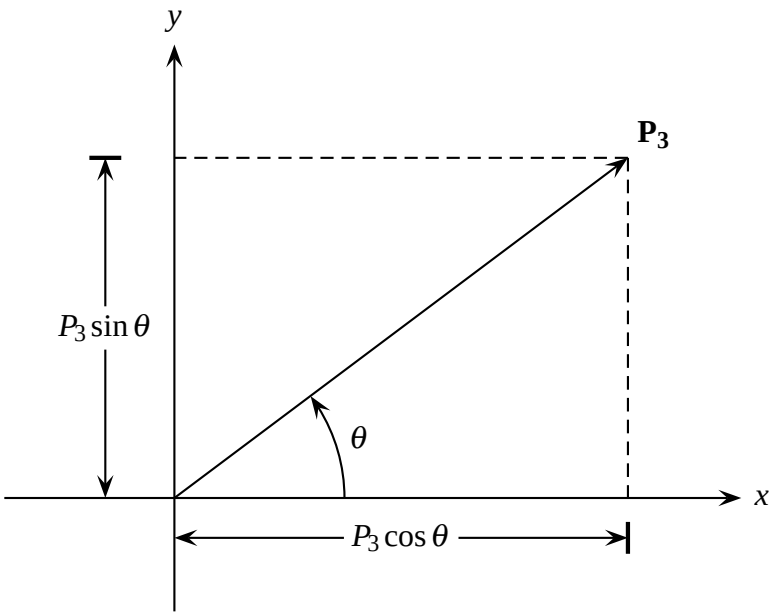


Figure 12.2: The component of $\mathbf{P_3}$ in the y-direction has a magnitude of $P_3 \sin\theta$, and the component of $\mathbf{P_3}$ in the x-direction has a magnitude of $P_3 \cos\theta$.

turn, the two vector components of the resultant vector. We then add these total x- and y-vector components together, using the parallelogram method, to find the resultant of the two original vectors. Let's see how this is done by looking at Figure 12.3.

In Figure 12.3, the two original vectors, the vectors to be added together, are labeled $\mathbf{P_3}$ and $\mathbf{P_4}$. Vector $\mathbf{P_3}$ forms angle θ (black) with the x-axis. Vector $\mathbf{P_4}$ forms angle ϕ (gray) with the same axis. Solid black is used to show $\mathbf{P_3}$, its x-component ($P\cos\theta$), and its y-component ($P\sin\theta$). Vector $\mathbf{P_4}$, its x-component ($P_4\cos\phi$), and its y-component ($P_4\sin\phi$), are shown in gray. The magnitude of the total vector in the x-direction is labeled P_x and the magnitude of the total vector in the y-direction vector is labeled P_y. The total vector is shown with a dashed line. In Figure 12.3, the vectors have the following properties:

Quantity	Black	Gray	Dashed
Magnitude	$P_3 = 15$	$P_4 = 20$	
Angle	$\theta = 30^\circ$	$\phi = 45^\circ$	
x-component	$P_3\cos\theta$	$P_4\cos\phi$	P_x
y-component	$P_3\sin\theta$	$P_4\sin\phi$	P_y

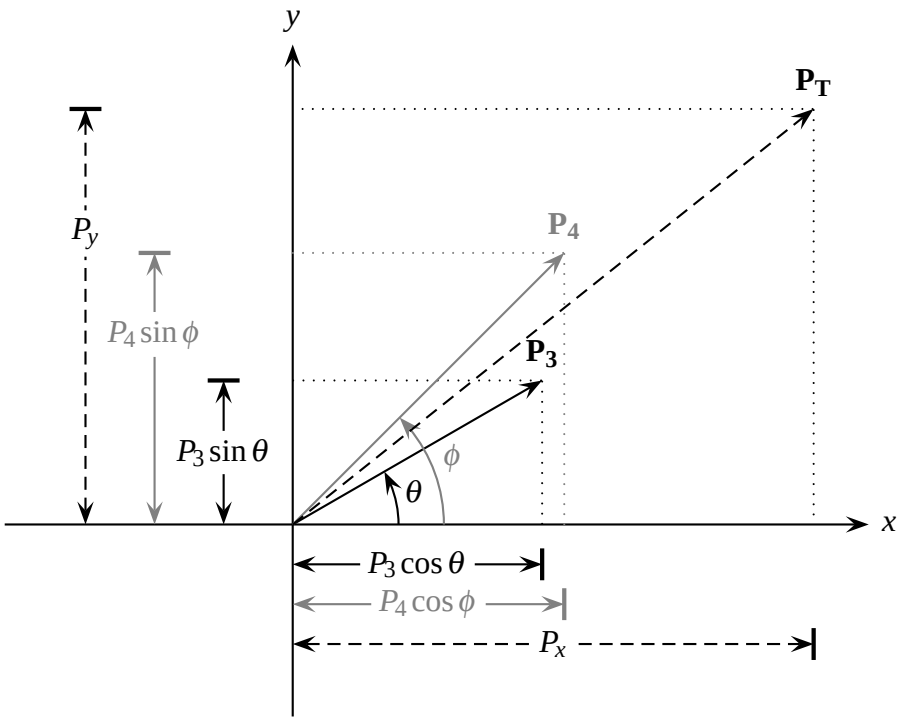


Figure 12.3: A diagram showing how to add two vectors together. This diagram is to scale with 1 inch being equal to 10 units.

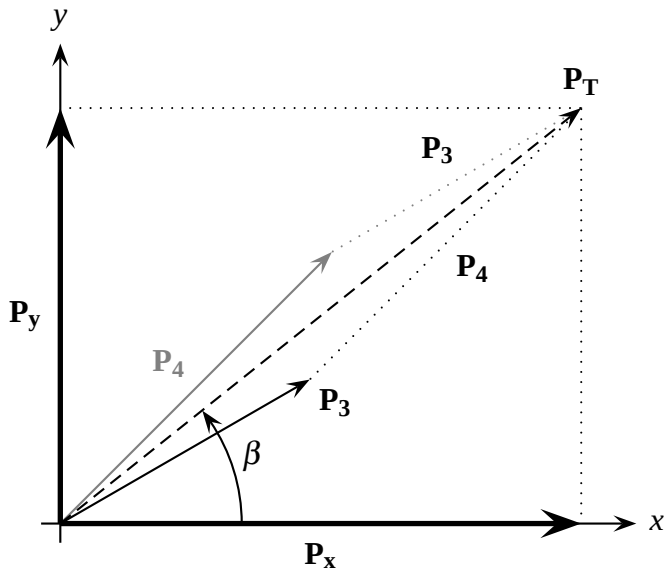


Figure 12.4: The graphical solution of the vector addition problem shown in Figure 12.3. This diagram illustrates that the vector sum of $\mathbf{P_3}$ and $\mathbf{P_4}$ is $\mathbf{P_T}$. By adding $\mathbf{P_3}$ and $\mathbf{P_4}$ together, we have accounted for both the magnitude of the vectors and the direction of the vectors.

Adding P_x and P_y in a vector sense gives the total, or resultant, vector, $\mathbf{P_T}$, as illustrated in Figure 12.4. Figure 12.4 also illustrates the use of the parallelogram method for adding vectors. Note the dotted lines on the diagram. The dotted line labeled $\mathbf{P_3}$ connects the head of vector $\mathbf{P_4}$ to the head of vector $\mathbf{P_T}$. This dotted line has the same magnitude and direction as $\mathbf{P_3}$, but is attached to the head of $\mathbf{P_4}$ rather than to the origin. The dotted line labeled $\mathbf{P_3}$ is thus parallel to vector $\mathbf{P_3}$. In the same way, the dotted line labeled $\mathbf{P_4}$ connects the head of vector $\mathbf{P_3}$ with the head of $\mathbf{P_T}$. It is parallel to vector $\mathbf{P_4}$ and has the same magnitude and direction as vector $\mathbf{P_4}$.

To determine the resultant vector mathematically, we will need to find the sine and cosine values for angles θ and ϕ to use in our calculations of the *x*- and *y*-components of the total vector.

Angle	Sine	Cosine
$\theta = 30^\circ$	0.500	0.866
$\phi = 45^\circ$	0.707	0.707

Calculate the total *x*-direction component:

$$P_x = P_3 \cos\theta + P_4 \cos\phi$$
$$P_x = 15(0.866) + 20(0.707)$$
$$P_x = 27.13$$

Calculate the total *y*-direction component:

$$P_y = P_3 \sin\theta + P_4 \sin\phi$$
$$P_y = 15(0.500) + 20(0.707)$$
$$P_y = 21.64$$

Now that we have calculated the magnitude of the components for the total, or resultant, vector, we can determine the resultant vector's magnitude and angle. The magnitude is determined from the

Pythagorean Theorem:

$$P_T = ||\mathbf{P_T}|| = \sqrt{P_x^2 + P_y^2}$$
$$P_T = \sqrt{(27.13)^2 + (21.64)^2}$$
$$P_T = 34.7$$

The angle, β, that $\mathbf{P_T}$ makes with the x-axis is calculated using the tangent function:

$$\begin{aligned}\beta &= \tan^{-1}\left(\frac{P_y}{P_x}\right) \\ &= \tan^{-1}\left(\frac{21.64}{27.13}\right) \\ &= \tan^{-1}(0.797) \\ &= 38.5^\circ\end{aligned}$$

Keep in mind that many calculators will give an inverse tangent between –90 and 90. Therefore, care must be taken to know in which quadrant the vector lies.

12.4.1.3 Collision Analysis Using Vector Diagrams

From Section 11.2 we have Eq. 11.9, the general equation for COLM:

$$\boxed{m_1\mathbf{v_1} + m_2\mathbf{v_2} = m_1\mathbf{v_3} + m_2\mathbf{v_4}} \tag{12.1}$$

Since $m = \frac{w}{g}$, it can be seen that g appears in each element of the momentum equation. Therefore, g can be cancelled out of each element of Eq. 12.1, leaving the equation in the form:

$$w_1v_1 + w_2v_2 = w_1v_3 + w_2v_4 \tag{12.2}$$

Over the last several pages, we have examined the addition of two vectors. These vectors may represent the post-impact momentum of two vehicles that have just been involved in a two-dimensional

collision, from which the vehicles departed at an angle. If this is the case, then $\mathbf{P_3} = m_1\mathbf{v_3}$ and $\mathbf{P_4} = m_2\mathbf{v_4}$. Vector $\mathbf{P_T}$ represents the total momentum of the system, or

$$\mathbf{P_T} = m_1\mathbf{v_3} + m_2\mathbf{v_4}$$

When momentum is conserved, vector $\mathbf{P_T}$ represents the total momentum of the system both before and after impact. Thus, if we know the value of $\mathbf{P_T}$ after the impact, we know the total momentum of the system. If we know the approach direction of each vehicle, we may then resolve $\mathbf{P_T}$ into components that are collinear with these approach directions. This will allow us to calculate the magnitudes of the velocities of the approaching vehicles. As we will see, we will also be able to calculate the change in velocity that each vehicle undergoes in the collision. This change will be in magnitude and in direction.

Note that we have defined a coordinate system for use in solving problems of this type, a polar coordinate system. We place the tails of the departure vectors on the origin. We measure all angles counterclockwise with respect to the *x*-axis. In order to handle the problem when it is expanded to show the approach vectors of the vehicles (or any other colliding bodies), we offer the following methodology:

a.) Choose the approach of one vehicle to be collinear with the *x*-axis. Its direction will be toward zero degrees. This simplifies the analysis greatly, as we need not concern ourselves with either the rotation nor the translation of the coordinate system.

b.) The vehicle chosen to be on the *x*-axis will be called vehicle #1 for the momentum analysis, regardless of what unit number it has in the crash report. The other vehicle will therefore be called vehicle #2.

c.) The tails of the approach vectors will be placed on the origin of the coordinate system. The approach vectors will thus be heading in the direction the vehicle would go if a collision did not alter the paths of the vehicles.

d.) Departure angles must be chosen with care. The departure angles must fall into an envelope formed by the two approach vectors. There is no system momentum that would cause the vehicles to depart outside this envelope. We will define an *impact circle* that will assist us in determining post-impact directions.

e.) As a matter of convention, the vehicle that is collinear with the *x*-axis, vehicle #1, will be identified with the subscript 1 upon approach and the subscript 3 upon departure. Likewise,

the momentum of the vehicle that has some *y*-axis momentum, vehicle #2, will be denoted with the subscript 2 upon approach and the subscript 4 upon departure. In this regard, $\mathbf{P_1} + \mathbf{P_2} = \mathbf{P_3} + \mathbf{P_4}$.

f.) Again as a matter of convention, we will name the departure angle of $\mathbf{P_3}$ "theta" (θ). The departure angle of $\mathbf{P_4}$ will be "phi" (ϕ). The approach angle of $\mathbf{P_2}$ will be "psi" (ψ). As the approach of $\mathbf{P_1}$ is always collinear with the *x*-axis, $\mathbf{P_1}$ will never have any *y*-axis momentum. The approach angle will be 0°. This angle need not be named, but may be put numerically into any subsequent calculation. This angle is sometimes referred to as "alpha" (α) in other literature.

g.) We will use momentum units of lb-mph in the US system and kg-kph in the SI system. We may do this because $m = \frac{w}{g}$ and $v = 1.466S$ (SI: $v = \frac{S}{3.6}$). Thus, m will have units of either lb or kg, and v will have units of mph or kph.

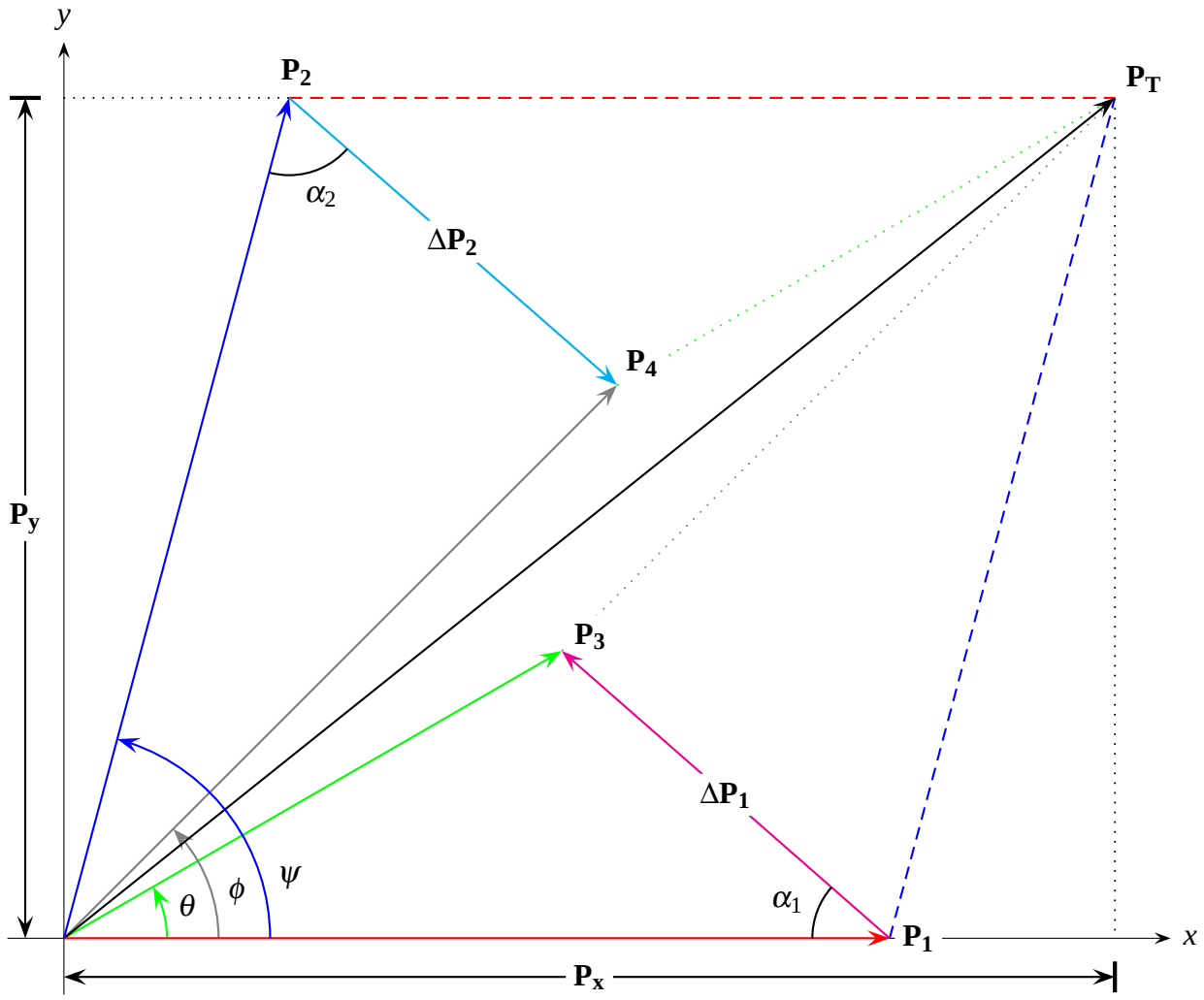
From the knowledge we have gained thus far, we may possibly see that there are several other methods for solving a momentum problem. However, the system just outlined, known as the 360° method, will provide uniformity with relatively simple bookkeeping requirements.

Having these points in mind, let us examine the diagram in Figure 12.5 on page 12-19.

Let's define each vector and angle that has been plotted. Then we will present three different, but related, methods for the analysis of this and similar diagrams.

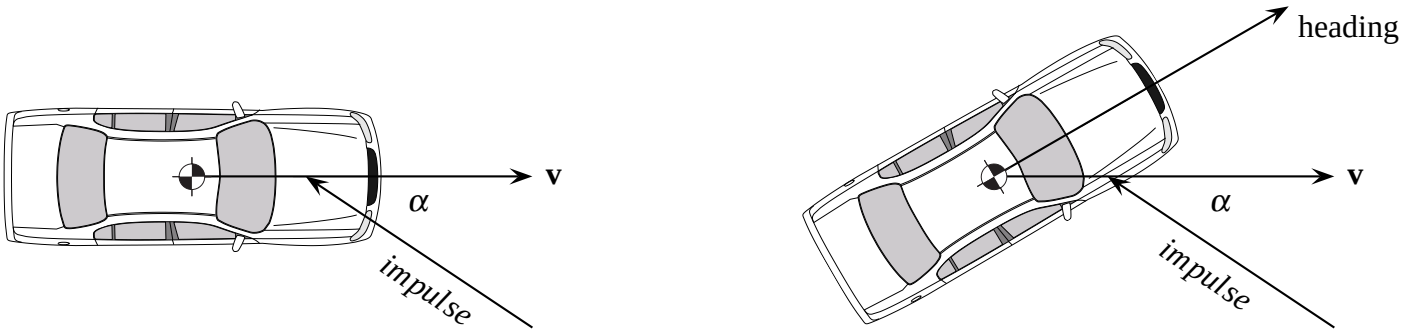
Vectors $\mathbf{P_3}$ and $\mathbf{P_4}$ represent the post-impact momentum vectors of vehicles #1 and #2. Vector $\mathbf{P_T}$ represents the total linear momentum of the system, both before and after impact. Vectors $\mathbf{P_1}$ and $\mathbf{P_2}$ represent the approach linear momentum vectors of vehicles #1 and #2. Vectors $\mathbf{P}_x$ and $\mathbf{P_y}$ represent the total *x*-axis and *y*-axis momentum vectors. Angles θ, ϕ, and ψ have already been discussed on the current page.

Vectors $\Delta\mathbf{P_1}$ and $\Delta\mathbf{P_2}$ represent the change in linear momentum of vehicles #1 and #2. As such, $\Delta\mathbf{P_1}$ and $\Delta\mathbf{P_2}$ each represent the *impulse* that acts on the vehicle designated by the subscript 1 or 2. This impulse acts through time and results in a force. This force is essentially the vector sum of all the forces acting on the vehicle from the moment of first contact through separation. If the vehicles do not separate, then this force acts from the moment of first contact until such time as a state of equilibrium exists between the two vehicles and they are no longer interacting with each other. We call this force the *Principal Direction of Force,* or PDOF. Recall Newton's Third Law that tells us $\Delta\mathbf{P_1}$ and $\Delta\mathbf{P_2}$ must have the same magnitude and also must be opposite in direction. Compare these two vectors in Figure 12.5, do they have the same magnitude and are they in opposite directions?



Scale: 1 inch = 42,000 units

Figure 12.5: A vector diagram solution to a collision using the conservation of linear momentum. To determine speeds, the length of the appropriate vector must be multiplied by the scale and then divided by the mass or weight of the vehicle.



(a) The vehicle's heading is collinear with its velocity vector.

(b) The vehicle's heading is *not* collinear with its velocity vector.

Figure 12.6: The center of mass of a vehicle is not always travelling in the same direction that the vehicle is pointing.

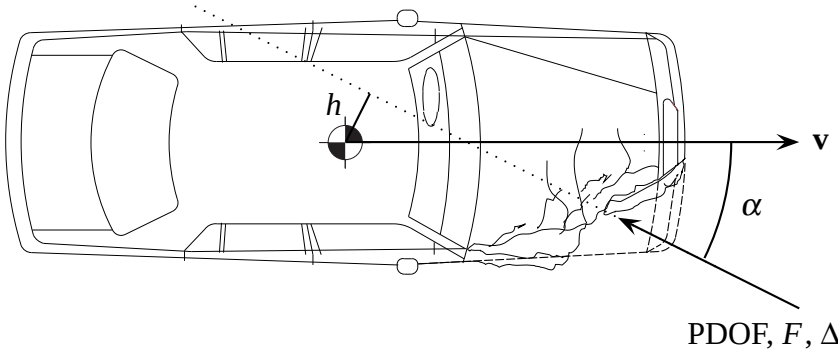


Figure 12.7: The solid, irregular lines define the contour of the damaged vehicle, and the dashed lines represent the undamaged profile of the vehicle. The damaged portion of the vehicle is the area enclosed by the solid irregular lines and the dashed lines.

Angles α_1 and α_2 represent the angles that the PDOF makes with respect to the pre-impact momentum (velocity) vectors of vehicles #1 and #2. These velocity vectors are referenced to the center of mass of each vehicle. If the vehicle is pointed in the same direction as the center of mass is traveling, then the heading and velocity vector will be collinear. If the vehicle is in some sort of rotated situation (the headlights are not pointing in the same direction as its velocity vector), the heading and the velocity vector will not be collinear. Figure 12.6 will help to illustrate this.

The PDOF acts upon the vehicle at the centroid of the damage area. A centroid may be thought of as the center of mass of a geometric figure, even though that figure may be massless. Figure 12.7 will help to show this.

Absent any constraints, the vehicle will want to rotate around its center of mass. In this particular example, the PDOF acts on lever arm h, and will cause a counterclockwise rotation. If the PDOF

acts through the center of mass, there will be no rotation. Clearly, if the PDOF passes on the other side of the center of mass, the vehicle will rotate clockwise.

Knowing the angle at which the PDOF acts on a vehicle may help us determine who was driving, should this be an issue. Determination of the driver is also assisted by the study of occupant motion inside the vehicle. This is called *occupant kinematics*. In occupant kinematics, we concern ourselves with the motion of the occupants with reference to the vehicle as opposed to their motion with reference to the ground. We then relate occupant motion inside the vehicle to the PDOF. It can be simplistically stated that the vehicle's occupants will tend to move toward and parallel to the PDOF. Thus, by matching occupant impact evidence gathered from the interior of the vehicle to the occupants of the vehicle, we can place those occupants, among them the driver, in their respective positions at the time of impact.

Continuing with our analysis of the vector diagram, let's examine two methods that we can use to find a solution to this momentum problem. We already know the values for $\mathbf{P_3}$, $\mathbf{P_4}$ θ, ϕ, ψ, and the approach direction of $\mathbf{P_1}$. We are interested in determining the following magnitudes: v_1, v_2, Δv_1, Δv_2, as well as the two PDOF angles, α_1 and α_2. Clearly v_1 is related to P_1, and v_2 is related to P_2, since $P = mv$. Similarly, Δv_1 is related to ΔP_1 and Δv_2 is related to ΔP_2 by the following general relationship:

$$\Delta P = m\Delta v \tag{12.3}$$

$$\Delta v = \frac{\Delta P}{m} \tag{12.4}$$

12.4.1.4 Constructing a Vector Diagram

To solve a momentum problem graphically, we must make a scale diagram of the vectors and plot them on a coordinate system. These plotted vectors represent momentums. We then measure the length of each vector line and convert this length into units of momentum, whereupon we divide these units by the weight of the respective vehicle to get the corresponding velocity. PDOF angles α_1 and α_2 may be measured from our scale diagram. An outline of the graphical method follows.

a.) Collect data from the scene of the collision that will enable you to determine θ, ϕ, and ψ. Gather evidence that will enable you to calculate v_3 and v_4. Weigh each vehicle.

b.) Construct a polar coordinate system. Plot lines on the polar system corresponding to angles θ, ϕ, and ψ, as shown in Figure 12.8.

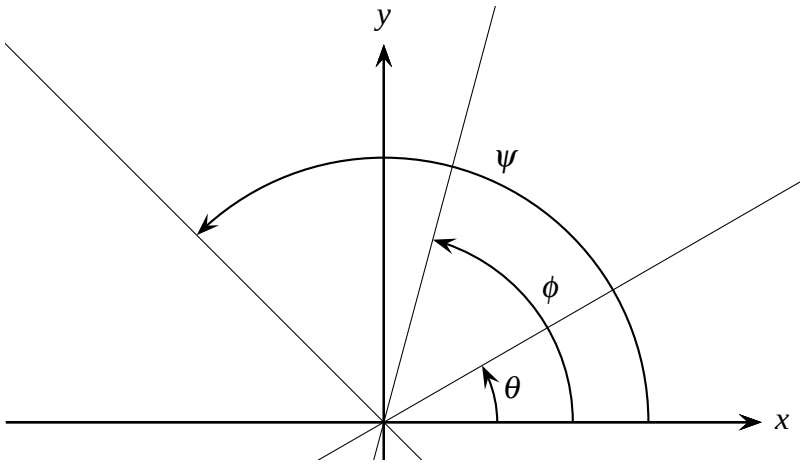


Figure 12.8: Plot the lines on the polar system corresponding to angles θ, ϕ, and ψ.

c.) Calculate v_3 and v_4, then calculate P_3 and P_4 using the equations $P_3 = w_1 v_3$ and $P_4 = w_2 v_4$. This will give results in units of momentum. Decide on a scale to use, for example, 1 inch = 40,000 units of momentum.

d.) Scale the line associated with angle θ by using the formula

$$l_3 = \frac{P_3}{scale}$$

Place an arrowhead a distance l_3, pointing away from the origin to denote the length of the vector. This line is the departure vector for vehicle #1. Label it $\mathbf{P_3}$, as shown in Figure 12.9.

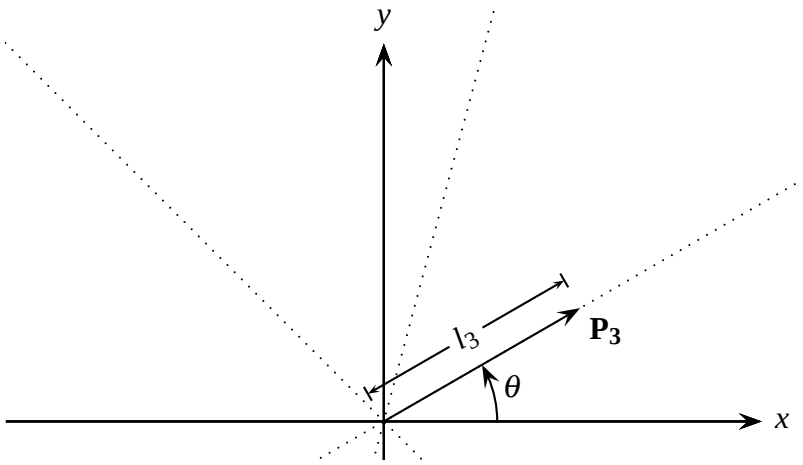


Figure 12.9: Scale the vector associated with angle θ.

e.) Scale the line associated with ϕ by using the formula

$$l_4 = \frac{P_4}{scale}$$

Again, place an arrowhead a distance l_4 pointing away from the origin to denote the length of the departure vector for vehicle #2. Label it $\mathbf{P_4}$, as shown in Figure 12.10.

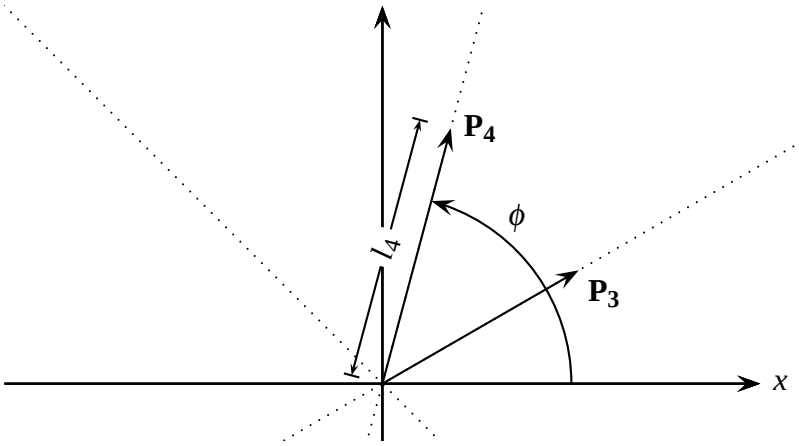


Figure 12.10: Scale the vector associated with angle ϕ.

f.) Draw a line parallel to $\mathbf{P_3}$ from the head of $\mathbf{P_4}$, and draw a second line parallel to $\mathbf{P_4}$ from the head of $\mathbf{P_3}$. The total momentum vector, $\mathbf{P_T}$, can be found by connecting the origin with the intersection of these two lines, as shown in Figure 12.11.

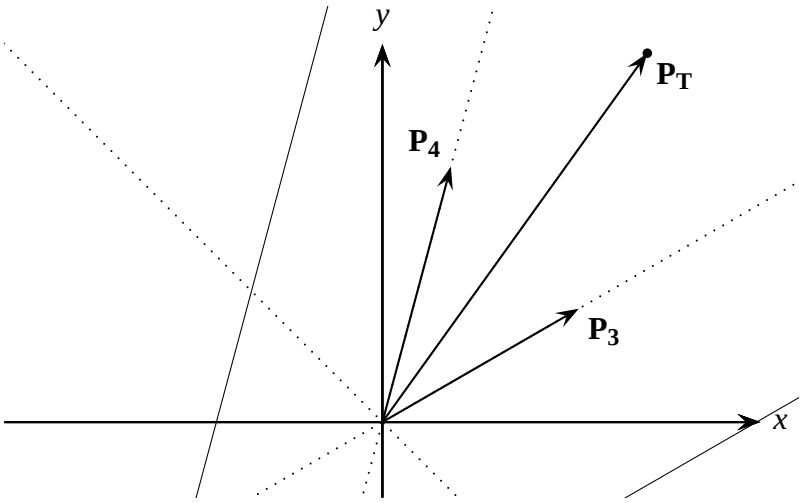


Figure 12.11: The total momentum is determined by the parallelogram method for adding vectors.

g.) Draw a construction line parallel to the *x*-axis from the head of $\mathbf{P_T}$. This line will intersect the line we have drawn at angle ψ. The intersection point defines the vector $\mathbf{P_2}$, the approach vector of vehicle #2. Measure the length of $\mathbf{P_2}$ and calculate the equivalent amount of momentum by using the equation

$$P_2 = l_2(scale)$$

Then calculate the approach velocity of vehicle #2 using the equation

$$v_2 = \frac{P_2}{w_2}$$

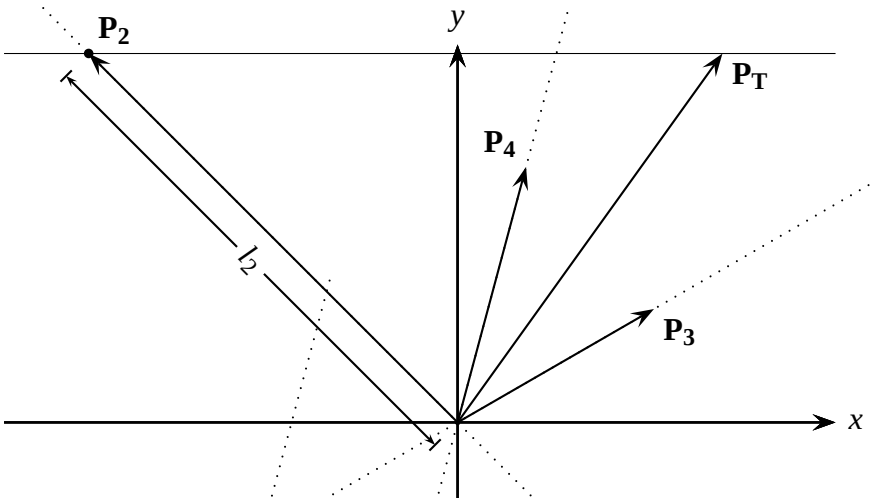


Figure 12.12: The approach vector of vehicle #2 is determined by the *y*-component of the total momentum vector.

h.) Draw a line parallel to the vector $\mathbf{P_2}$ from the head of $\mathbf{P_T}$ to the *x*-axis. The point at which this line intersects the *x*-axis defines the value $\mathbf{P_1}$, the approach vector of vehicle #1. Measure the length of $\mathbf{P_1}$ and calculate the equivalent amount of momentum by using the equation

$$P_1 = l_1(scale)$$

Then calculate the approach velocity of vehicle #1 using the equation

$$v_1 = \frac{P_1}{w_1}$$

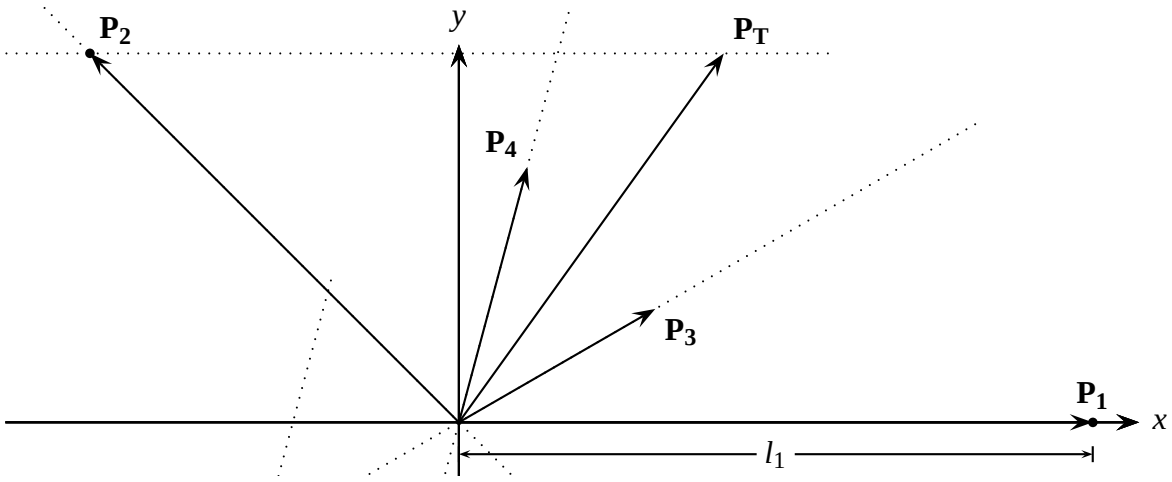


Figure 12.13: The approach vector of vehicle #1 is determined by the x-component of the total momentum vector.

i.) Draw a vector from the head of $\mathbf{P_1}$ to the head of $\mathbf{P_3}$. This arrow will define the vector $\blacksquare\mathbf{P_1}$. Draw a vector from the head of $\mathbf{P_2}$ to the head of $\mathbf{P_4}$. This line will define $\blacksquare\mathbf{P_2}$. These lines should be equal in length and opposite in direction, as shown in Figure 12.14 on page 12-26.

j.) Determine the magnitude of $\blacksquare\mathbf{P_1} = \blacksquare\mathbf{P_2}$ by multiplying the length of these vectors l_Δ by the scale.

$$\Delta v_1 = \frac{\Delta P_1}{w_1} \qquad \text{and} \qquad \Delta v_2 = \frac{\Delta P_2}{w_2}$$

k.) The principal direction of force (PDOF) angles α_1 and α_2 can be measured from the scale diagram, as shown in Figure 12.14 on page 12-26.

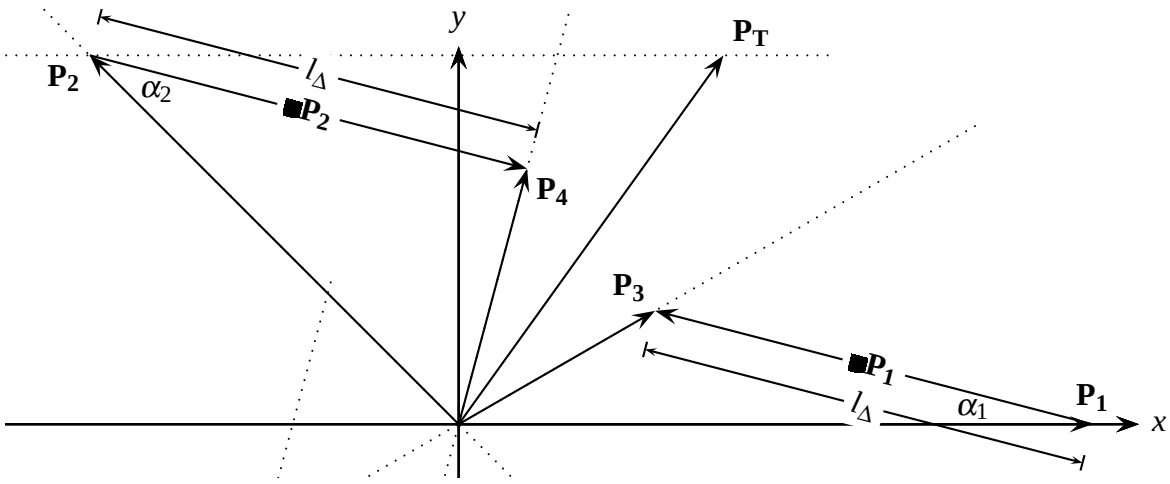


Figure 12.14: The magnitude and directions of the impulse vectors can be measured from the scale diagram.

12.4.2 Mathematical Analysis

12.4.2.1 Derivation of Basic Equations

In the previous section we solved a two-vehicle, two-dimensional collision by creating a vector diagram using a scale drawing, protractor, and a ruler. The same solution can be obtained using both geometry and trigonometry to analyze the vectors and angles. We will use our general linear momentum diagram shown in Figure 12.5 on page 12-19 to help us in the analysis.

Working with y-direction vectors

The *y*-direction components of the total momentum vector must be equal before and after the collision. Recall that we defined our coordinate system so that vehicle #1 has no component in the *y*-direction for the incoming momentum. This means that only vehicle #2 has a *y*-momentum component that has to equal the momentum in the *y*-direction after the crash.

$$P_2 \sin\psi = P_3 \sin\theta + P_4 \sin\phi \tag{12.5}$$

Divide through by $\sin\psi$:

$$P_2 = \frac{P_3 \sin\theta}{\sin\psi} + \frac{P_4 \sin\phi}{\sin\psi} \tag{12.6}$$

From the general linear momentum equation (Eq. (11.3)):

$$P = mv = \frac{wv}{g} \tag{12.7}$$

Substitute:

$$\frac{w_2 v_2}{g} = \frac{w_1 v_3 \sin\theta}{g \sin\psi} + \frac{w_2 v_4 \sin\phi}{g \sin\psi} \tag{12.8}$$

Multiply through by g:

$$w_2 v_2 = \frac{w_1 v_3 \sin\theta}{\sin\psi} + \frac{w_2 v_4 \sin\phi}{\sin\psi} \tag{12.9}$$

Divide through by w_2 :

$$\boxed{v_2 = \frac{w_1 v_3 \sin\theta}{w_2 \sin\psi} + \frac{v_4 \sin\phi}{\sin\psi}} \tag{12.10}$$

Working with x direction vectors

We know that the x-direction components of the momentum before and after the collision must be the same. This gives the following result:

$$P_1 \cos 0^\circ + P_2 \cos\psi = P_3 \cos\theta + P_4 \cos\phi \tag{12.11}$$

Substitute $\cos 0^\circ = 1$ and subtract $P_2 \cos\psi$ from both sides, yielding:

$$P_1 = P_3 \cos\theta + P_4 \cos\phi - P_2 \cos\psi \tag{12.12}$$

From the general linear momentum equation (Eq. (11.3)):

$$P = mv = \frac{wv}{g} \tag{12.13}$$

Substitute Eq. (12.13) into Eq. (12.12):

$$\frac{w_1 v_1}{g} = \frac{w_1 v_3 \cos\theta}{g} + \frac{w_2 v_4 \cos\phi}{g} - \frac{w_2 v_2 \cos\psi}{g} \tag{12.14}$$

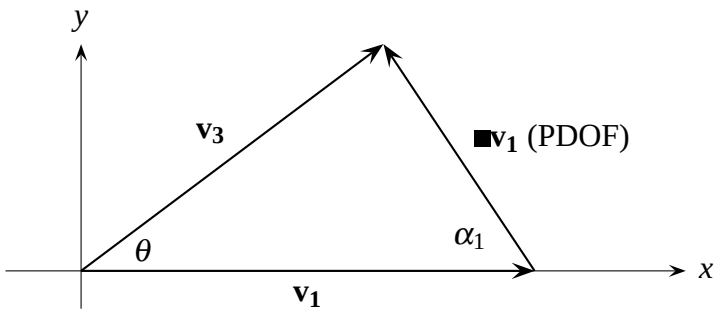


Figure 12.15: The triangle formed by the velocity vectors of vehicle #1 is, in general, oblique and the properties of a right triangle do not hold.

Multiply through by g:

$$w_1 v_1 = w_1 v_3 \cos\theta + w_2 v_4 \cos\phi - w_2 v_2 \cos\psi \tag{12.15}$$

Divide through by w_1, yielding:

$$\boxed{v_1 = v_3 \cos\theta + \frac{w_2 v_4 \cos\phi}{w_1} - \frac{w_2 v_2 \cos\psi}{w_1}} \tag{12.16}$$

Notice that Eq. (12.16) requires the use of v_2. This is why we compute v_2 from Eq. (12.10) first.

12.4.2.2 Derivation of the ■v and PDOF Equations

Now that we have derived equations to calculate v_1 and v_2, we will look for a method that enables us to calculate Δv_1 and Δv_2. We still need to determine PDOF angles α_1 and α_2. The part of Figure 12.5 that only deals with vehicle #1 is shown in Figure 12.15.

Even though the triangle in Figure 12.15 may not be a right triangle, the Law of Cosines and the Law of Sines still apply. (See Section **??** on page ?? for review of these laws).

The Law of Cosines states that if two sides and the included angle of a triangle are known, the third side can be calculated using the equation:

$$c = \sqrt{a^2 + b^2 - 2ab\cos\theta}$$

where a and b are the known sides, θ is the included angle, and c is the third side.

We see from Figure 12.15 that we know v_1, v_3, and θ. Therefore, $a = v_1$, $b = v_3$, and $c = \Delta v_1$.

Substituting:

$$\boxed{\Delta v_1 = \sqrt{v_1^2 + v_3^2 - 2v_1 v_3 \cos\theta}} \tag{12.17}$$

The Law of Sines states that if two angles and the included side of a triangle are known, the third side can be calculated using the relationship:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

However, because the Law of Sines is a relationship of ratios, we can use any pair to find one of the unknowns if we know the other three. We see from Figure 12.15 that we know Δv_1, v_3, and θ, and are trying to find α_1. Therefore, $a = \Delta v_1$, and $b = v_3$.

Substituting:

$$\frac{\Delta v_1}{\sin\theta} = \frac{v_3}{\sin\alpha_1}$$

Solving for α_1 yields:

$$\boxed{\alpha_1 = \sin^{-1}\left(\frac{v_3 \sin\theta}{\Delta v_1}\right)} \tag{12.18}$$

In a similar manner, we will develop our equations for Δv_2 and α_2. Begin with the diagram shown in Figure 12.16 on page 12-30 that corresponds to vehicle #2 in Figure 12.5.

Figure 12.5 has already illustrated that, in accordance with Newton's Third Law, $\blacksquare\mathbf{P_1}$ and $\blacksquare\mathbf{P_2}$ have the same magnitude and the opposite direction. As a side observation, an extension of the $\blacksquare\mathbf{v_2}$ line intersects the x-axis and forms angle α_1, the same angle as shown in Figures 12.5 and 12.15. Agreement of α_1 here with α_1 in those figures will provide us with a check on our work.

We will again apply the Law of Cosines:

$$\boxed{\Delta v_2 = \sqrt{v_2^2 + v_4^2 - 2v_2 v_4 \cos(\psi - \phi)}} \tag{12.19}$$

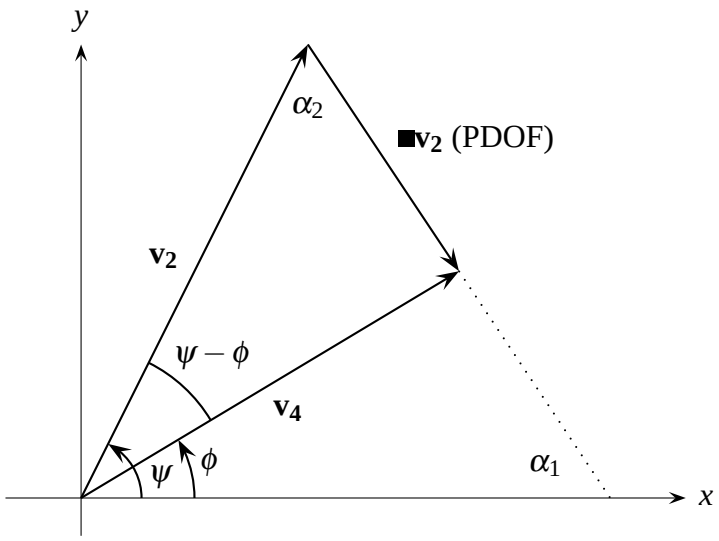


Figure 12.16: This is the triangle formed by the vectors of vehicle # 2. We know that due to Newton's Third Law, the magnitude ΔP_2 must be equal to ΔP_1, so $\Delta v_2 = \frac{w_1}{w_2}\Delta v_1$.

From the Law of Sines:

$$\boxed{\alpha_2 = \sin^{-1}\left(\frac{v_4 \sin(\psi - \phi)}{\Delta v_2}\right)} \tag{12.20}$$

The angles in a triangle must add up to 180°, so we can perform a check on our work:

$$\psi = 180^\circ - \alpha_1 - \alpha_2 \tag{12.21}$$

Example 12.4. Two vehicles are involved in a collision. The approach of vehicle #1 is toward 0°. The approach of vehicle #2 is toward 75°. The departure of vehicle #1 is at 30°. The departure of vehicle #2 is at 45°. $v_3 = 30$ mph (48 km/h). $v_4 = 35$ mph (56 km/h). $w_1 = 3000$ lb (1363 kg). $w_2 = 3500$ lb (1591 kg). Calculate v_1, v_2, Δv_1, Δv_2, α_1, and α_2.

Solution

Begin by calculating v_2 with Eq. (12.10):

$$v_2 = \frac{w_1 v_3 \sin\theta}{w_2 \sin\psi} + \frac{v_4 \sin\phi}{\sin\psi}$$

US

$$v_2 = \frac{3000(30)\sin 30}{3500\sin 75} + \frac{(35)\sin 45}{\sin 75}$$
$$= 38.96\,\text{mph}$$

SI

$$v_2 = \frac{1363(48)\sin 30}{1591\sin 75} + \frac{(56)\sin 45}{\sin 75}$$
$$v_2 = 62.33\,\text{kph}$$

Next calculate v_1 using Eq. (12.16):

$$v_1 = v_3\cos\theta + \frac{w_2 v_4 \cos\phi}{w_1} - \frac{w_2 v_2 \cos\psi}{w_1}$$

$$v_1 = 30(0.866) + \frac{3500(35)(.707)}{3000} - \frac{3500(38.96)(0.258)}{3000}$$

$$v_1 = 25.98 + 28.86 - 11.72$$

$$v_1 = 43.12\,\text{mph}$$

$$v_1 = 48(0.866) + \frac{1591(56)(.707)}{1363} - \frac{1591(62.33)(0.258)}{1363}$$

$$v_1 = 41.56 + 46.21 - 18.77$$

$$v_1 = 69.01\,\text{kph}$$

Now calculate Δv_1 using Eq. (12.17):

$$\Delta v_1 = \sqrt{v_1^2 + v_3^2 - 2v_1 v_3 \cos\theta}$$

US	SI
$\Delta v_1 = \sqrt{43.12^2 + 30^2 - 2(43.12)(30)(0.866)}$	$\Delta v_1 = \sqrt{69.01^2 + 48^2 - 2(69.01)(48)(0.866)}$
$\Delta v_1 = \sqrt{2759.33 - 2240.51}$	$\Delta v_1 = \sqrt{7066.38 - 5737.21}$
$\Delta v_1 = 22.77\,\text{mph}$	$\Delta v_1 = 36.45\,\text{kph}$

Next calculate Δv_2 using Eq. (12.19) when $\cos(\psi - \phi) = \cos(75 - 45) = \cos(30) = 0.866$:

$$\Delta v_2 = \sqrt{v_2^2 + v_4^2 - 2v_2 v_4 \cos(\psi - \phi)}$$

$\Delta v_2 = \sqrt{38.96^2 + 35^2 - 2(38.96)(35)(0.866)}$	$\Delta v_2 = \sqrt{62.33^2 + 56^2 - 2(62.33)(56)(0.866)}$
$\Delta v_2 = \sqrt{2742.88 - 2361.75}$	$\Delta v_2 = \sqrt{7021.02 - 6045.51}$
$\Delta v_2 = 19.52\,\text{mph}$	$\Delta v_2 = 31.23\,\text{kph}$

Calculate the PDOF angle for vehicle #1 using Eq. (12.18):

$$\alpha_1 = \sin^{-1}\left(\frac{v_3 \sin\theta}{\Delta v_1}\right)$$

$$\alpha_1 = \sin^{-1}\left(\frac{30(0.500)}{22.77}\right)$$
$$\alpha_1 = \sin^{-1}(0.658)$$
$$\alpha_1 = 41^\circ$$

$$\alpha_1 = \sin^{-1}\left(\frac{48(0.500)}{36.45}\right)$$
$$\alpha_1 = \sin^{-1}(0.658)$$
$$\alpha_1 = 41^\circ$$

Calculate the PDOF angle for vehicle #2 using Eq. (12.20):

$$\alpha_2 = \sin^{-1}\left(\frac{v_4 \sin(\psi - \phi)}{\Delta v_2}\right)$$

US

$$\alpha_2 = \sin^{-1}\left(\frac{35(0.500)}{19.52}\right)$$
$$\alpha_2 = \sin^{-1}(0.896)$$
$$\alpha_2 = 64^\circ$$

SI

$$\alpha_1 = \sin^{-1}\left(\frac{56(0.500)}{31.23}\right)$$
$$\alpha_1 = \sin^{-1}(0.896)$$
$$\alpha_1 = 64^\circ$$

We can check our PDOF angles using Eq. (12.21):

$$\psi = 180^\circ - \alpha_1 - \alpha_2$$
$$\psi = 180^\circ - 41^\circ - 64^\circ$$
$$75^\circ = 75^\circ$$

Check!

We can also determine the amount of energy lost in this collision. Clearly not all of the pre-impact kinetic energy was used to deform the two vehicles. That which was not used up in the collision

was dissipated in the post-impact trajectory. The energy-lost calculation is simple:

$$KE_{in} = KE_{out} + KE_{lost}$$

So:

$$KE_{lost} = KE_{in} - KE_{out}$$

$$KE_{lost} = \left(\frac{w_1 S_1^2}{30} + \frac{w_2 S_2^2}{30}\right) - \left(\frac{w_1 S_3^2}{30} + \frac{w_2 S_4^2}{30}\right)$$

$$KE_{lost} = \left(\frac{3000(43.12)^2}{30} + \frac{3500(38.96)^2}{30}\right) - \left(\frac{3000(30)^2}{30} + \frac{3500(35)^2}{30}\right)$$

$$KE_{lost} = (185,933 + 177,086) - (90,000 + 142,916)$$

$$KE_{lost} = 130,103\,\text{ft-lb}$$

$$KE_{lost} = \left(\frac{m_1 S_1^2}{26} + \frac{m_2 S_2^2}{26}\right) - \left(\frac{m_1 S_3^2}{26} + \frac{m_2 S_4^2}{26}\right)$$

$$KE_{lost} = \left(\frac{1363(69.01)^2}{26} + \frac{1591(62.33)^2}{26}\right) - \left(\frac{1363(48)^2}{26} + \frac{1591(56)^2}{26}\right)$$

$$KE_{lost} = (249,658 + 237,733) - (120,782 + 191,899)$$

$$KE_{lost} = 174,710\,\text{J}$$

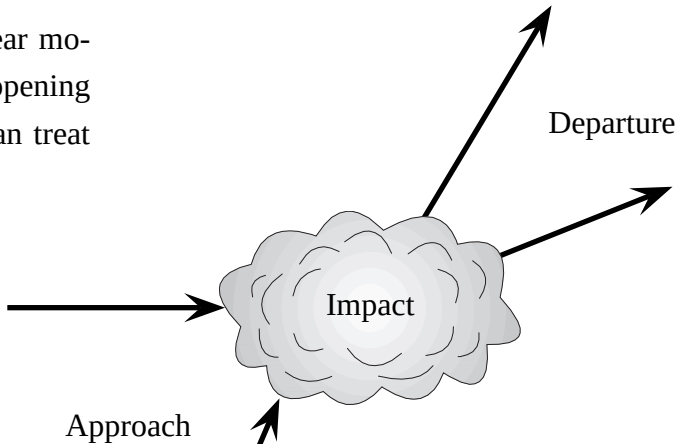
By the way, the vector diagram in Figure 12.5 is the vector diagram solution to Example 12.4. See if the angles and lengths yield the same values just calculated.

12.5 Evidence Required for COLM Analysis

12.5.1 Determining Pre- and Post-Impact Directions Using the Impact Circle

The concept of the *impact circle* was first presented by Dr. Gordon Bigg during his presentation, "Momentum - Facts and Myths" at the 1998 IPTM Special Problems seminar. This has proven to be a useful method for determining post-impact directions for two-dimensional collisions. The impact circle is a region in space and time where the collision forces act upon the vehicles.

The analysis using conservation of linear momentum does not consider what is happening to the vehicles during the crash. We can treat



CONSERVATION OF LINEAR MOMENTUM EQUATIONS

a.)

$$v_2 = \frac{w_1 v_3 \sin\theta}{w_2 \sin\psi} + \frac{v_4 \sin\phi}{\sin\psi}$$

b.)

$$v_1 = v_3 \cos\theta + \frac{w_2 v_4 \cos\phi}{w_1} - \frac{w_2 v_2 \cos\psi}{w_1}$$

c.)

$$\Delta v_1 = \sqrt{v_1^2 + v_3^2 - 2 v_1 v_3 \cos\theta}$$

d.)

$$\Delta v_2 = \sqrt{v_2^2 + v_4^2 - 2 v_2 v_4 \cos(\psi - \phi)}$$

e.)

$$\alpha_1 = \sin^{-1}\left(\frac{v_3 \sin\theta}{\Delta v_1}\right)$$

f.)

$$\alpha_2 = \sin^{-1}\left(\frac{v_4 \sin(\psi - \phi)}{\Delta v_2}\right)$$

g.)

$$\psi \approx 180^\circ - \alpha_1 - \alpha_2$$

Note 1: $\Delta v_1 = \Delta v_2$ only if $w_1 = w_2$.
Note 2: Equation 7 should agree within a degree or two of the measured angle ψ.

Table 12.5: Seven equations that can be used to investigate collisions using conservation of linear momentum.

the collision as a "black box" in which the vehicles are interacting. The impact circle is a circle that can be drawn around the impact area where the vehicles are interacting. The departure angles are determined as those angles at which the vehicles are moving when they stop interacting with collision forces and thus exit the impact circle. Usually, we use the term "point of impact," which should more correctly be the "area of impact" because a collision takes some time and one specific point is not large enough to completely describe the collision.

The impact circle includes secondary slaps. For example, a secondary slap may occur when a car impacts the front end of a pickup with some offset configuration and the rear of the pickup spins around to hit the car. There is no need to treat these secondary impacts separately because they are included in the impact circle. The principle of conservation of linear momentum states that the total linear momentum before the collision is equal to the total linear momentum after the collision. Momentum is exchanged during the collision. Nothing says that this exchange has to occur in one impact or two. As long as there is interaction between the vehicles (and no external forces enter the system), the impact circle should be drawn around that area. Remember that interaction is defined as exerting force. If two vehicles become entangled and move off as one, although they are still touching, they are not to be considered interacting with collision forces.

Look for evidence on the ground (road) leading up to the impact to determine the direction each vehicle was traveling at the moment of impact. Pre-impact skid marks are defining evidence, but are not always available. Pre-impact spins are also excellent evidence of direction. If the spin indicates a curving path into the collision, then the pre-impact direction is a line tangent to the curve at the moment of impact. In cases in which no driver action prior to impact is noted, it may well be that the vehicle was traveling parallel to the road. In cases in which one vehicle turns into the path of another, the direction is tangent to the turn path at the moment of collision. If a reasonable range of pre-impact directions cannot be established, then a COLM analysis is not appropriate.

Using the impact circle to determine post-impact directions is not difficult. Examine the crash scene for evidence of where on the ground the collision took place, such as gouge marks or tire scuffs. On a scale diagram of the crash scene, move the vehicles through the collision interaction(s) and determine where the collision forces have stopped acting. The post-impact direction is then tangent

to the path of the vehicle at that point. If reasonable post-impact directions cannot be established, then a COLM analysis is not appropriate.

The physical evidence should give an indication of the position of the vehicles both before and after the collision. These locations correspond to the boundary of the impact circle.

The measures of the approach and departure angles, ψ, θ, and ϕ, are then determined after one of the vehicles has been assigned to the *x*-axis. The approach and departure angles are then measured with respect to the *x*-axis.

12.5.2 Determining Vehicle Weights

As we have seen, the momentum possessed by a vehicle coming into or out of a collision is a function of both velocity and mass. Hence, we must determine as accurately as possible the weight of each vehicle both in and out of a collision. It is best to actually weigh the vehicles if this is possible, as the curb weight listed in any of several data bases will not include occupants or any cargo, luggage, tools, or whatever else may be in the vehicle. If it is not possible to weigh the vehicles, then we must do our best to add the weights of all occupants and other items that were in the vehicle to its curb weight. If vehicles break apart during collision, then we must determine the individual weights of each vehicle section.

12.5.3 Determining Post-Impact Speeds

Finding post-impact speed accurately is an important part of the overall COLM analysis. Post-impact speed, coupled with direction, gives us the post-impact velocity we need to do the analysis. A simple slide-to-stop analysis will probably not be sufficient for the determination of post-impact speeds in many real world crashes. If the vehicle rotated away from the impact, then a spin analysis as described in Chapter **??** would be appropriate. If the vehicle overturned, then we may determine drag factors from the references in Chapter **??**. Examine carefully how each vehicle got to its final rest position and then apply the appropriate speed analysis.

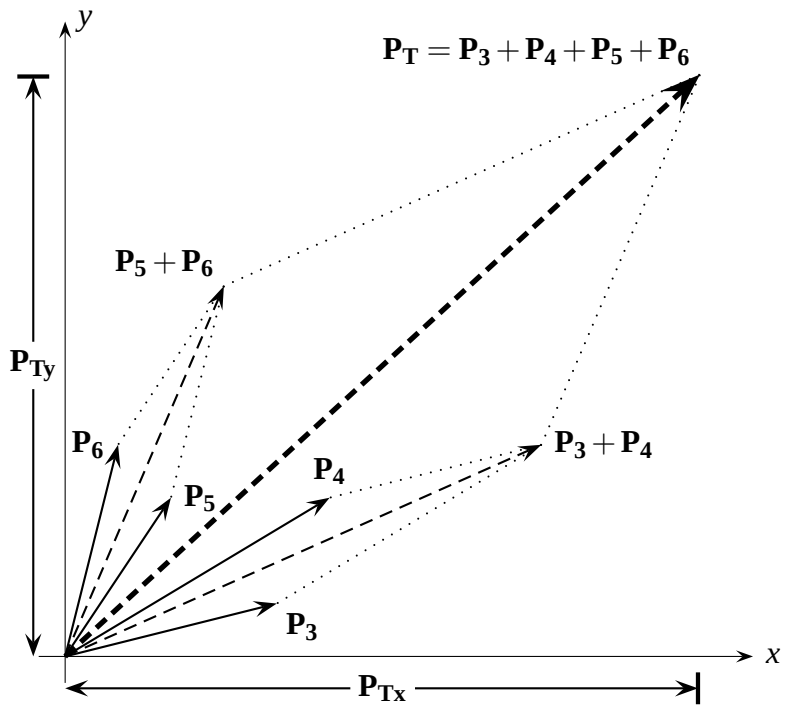


Figure 12.18: An example of a post-impact vector diagram for a collision between two units when they both break apart. The total momentum is the vector sum of all four parts.

12.6 Special Considerations and Limitations

12.6.1 Multiple Departure Analysis

Sometimes a crash will involve vehicles that break apart during the collision. This situation can still be solved using conservation of linear momentum. Recall from Section 11.2 that when the conservation of linear momentum applies, it does not matter how many objects are involved. If a vehicle breaks apart upon impact, these can be treated as two separate bodies, each with their own momentum. If the other vehicle remains intact, then the post-impact momentum is simply the sum of three vectors instead of two. The total incoming momentum must still be the same as the total outgoing momentum, so the analysis, after the total momentum has been determined, remains the same.

For example, let's say two vehicles collide and both break in half. Figure 12.18 shows a vector diagram constructed for four pieces coming out of the collision and only two coming in. Solving this problem might seem tricky at first, but if two vectors are added at a time, then the problem slowly becomes smaller until it is like a problem we have already seen. The biggest challenge is the bookkeeping of all the angles and symbols.

The momentum equations for the diagram in Figure 12.18 would look like:

$$v_2 = \frac{w_{1\,front} v_{3\,front} \sin\theta_{1\,front}}{w_2 \sin\psi} + \frac{w_{1rear} v_{3rear} \sin\theta_{1rear}}{w_2 \sin\psi} + \frac{w_{2\,front} v_{4\,front} \sin\phi_{2\,front}}{w_2 \sin\psi} + \frac{w_{2rear} v_{4rear} \sin\phi_{2rear}}{w_2 \sin\psi}$$

$$v_1 = \frac{w_{1\,front} v_{3\,front} \cos\theta_{1\,front}}{w_1} + \frac{w_{1rear} v_{3rear} \cos\theta_{1rear}}{w_1} + \frac{w_{2\,front} v_{4\,front} \cos\phi_{2\,front}}{w_1} + \frac{w_{2rear} v_{4rear} \cos\phi_{2rear}}{w_1} - \frac{w_2 v_2 \cos\psi}{w_1}$$

There are four distinct objects, each with its own distinct departure angle.

Sometimes a collision occurs that looks like a multiple departure momentum problem but is not. Consider a case in which a car slides sideways into a pole. The impact breaks the car into two pieces, each of which then travels to a final rest. It may be tempting for the unwary investigator to calculate the momentum of each piece and then relate that back to the impact speed of the car. However, this approach will result in a large error, significantly underestimating the impact speed of the car. The vector sum of the two momentum vectors going out of the collision represents the *post-impact* momentum of the vehicle after it has interacted with the pole. This analysis also ignores the significant collision impulse, which is external to the moving vehicle. Essentially, the pole is coupled to the earth, and a traditional COLM analysis does not apply. In such cases, we must account for the impulse from the immovable pole. Getting an accurate calculation of the collision impulse may not be possible in this case in which the vehicle is cut in two.

12.6.2 Multiple Collisions

Many times there may be more than two vehicles per crash. In these cases, it is necessary to determine the order of the crash sequence. If COLM analyses are applicable, then the *last* collision must be analyzed *first* and the first collision last. The reason for this is because the solution of the initial crash depends on the solution of the final crash. The solution of the final crash depends on the post-impact speeds and angles, which are determined from the physical evidence.

Since collision times are on the order of 0.100 seconds, it is very rare to have more than two vehicles collide at the same time. Sometimes, however, the order in which the crash happened may not be obvious.

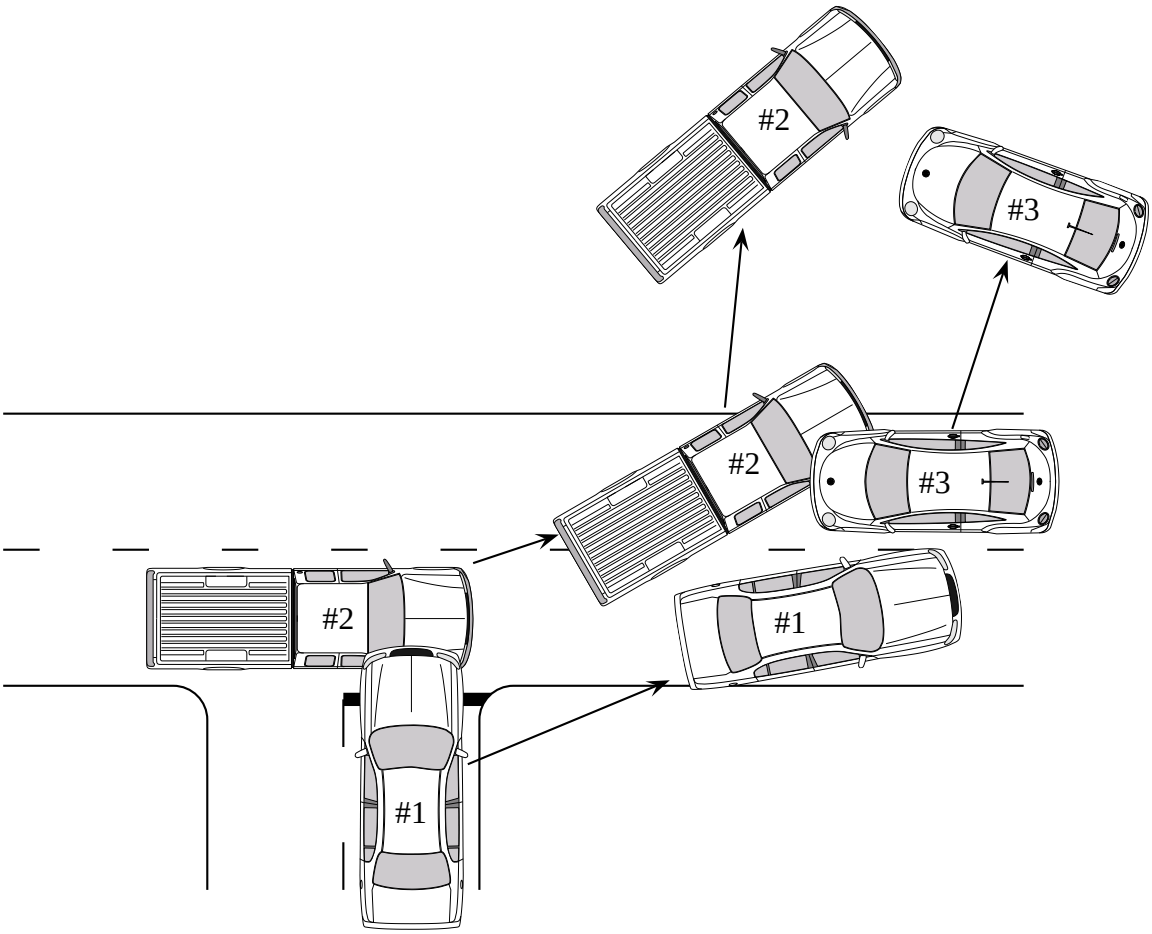


Figure 12.19: A crash scene in which vehicles #1 and #2 collide, then #2 and #3 collide. To solve this, analyze the last crash first.

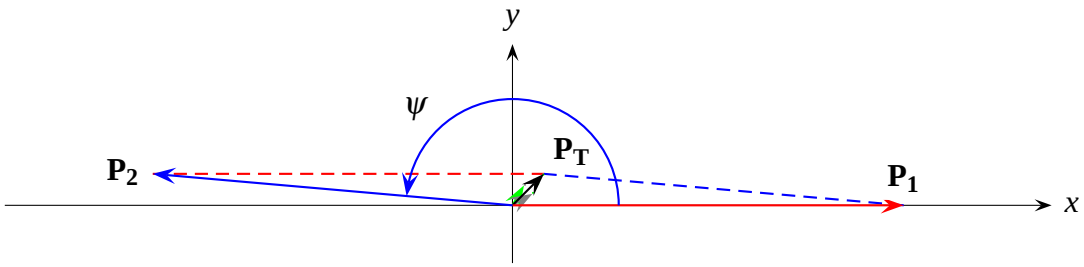
12.6.3 Low Speeds

A COLM analysis in the form presented in this chapter is not always appropriate for a low speed impact, such as that occurring in a parking lot or at a stop light when one vehicle rolls into another.

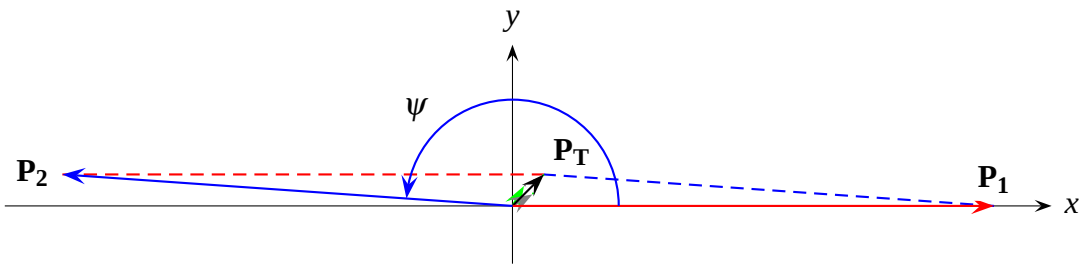
At speeds under 10 to 15 mph, restitution effects may be significant. In low speed crashes, collision forces may not be high enough to overcome the ever-present ground forces. In this case, the frictional force must be included as an impulse term on the momentum equation. This is discussed in Chapter 11.

12.6.4 Small Approach Angles

When collisions occur near head-on or near in-line, the determination of the angle ψ must be extremely accurate. The reason is because the sine of a small angle changes very quickly which may result in an inappropriate speed calculation for vehicle #2. This is best illustrated by the graphical solution to a 175° approach momentum problem, as shown in Figure 12.20. If the approach angles are 170 to 180 (0 to 10) degrees, then a collinear approach using a Damage–Momentum solution as outlined in Chapter **??** or simultaneous equations as explained in Chapter **??** may be more appropriate.



(a) The approach angle ψ is 175°.



(b) The approach angle ψ is 176°.

Figure 12.20: Two momentum diagrams in which the approach vectors of vehicle #2 differ by only 1°. If the scale is 1 inch = 100,000 units, then the difference in speeds for both vehicles is significant.

12.6.5 Large Weight Differences

There are two problems associated with COLM analysis when the mass ratio between the impacting objects gets large, on the order of 10 to 1 or higher. Depending on the orientation of the vehicles during the collision, there may be significant ground frictional forces from the larger vehicle that must be accounted for. This accounting, which will show up as an impulse term in the COLM equation, may be difficult to quantify. Secondly, a small change of the post-impact speed for the larger vehicle will result in a much larger change in the impact speed of the smaller vehicle. In such cases, the quality of the evidence is paramount. Even with the best of evidence, these variances may make the traditional COLM analysis unusable for determining the speed of the lighter object or vehicle.

Example 12.5. A 4500-lb (2041-kg) truck hits a 150-lb (68-kg) person crossing the street. The post impact skids of the truck are 50 ft (15.2 m) long and the pre-impact skids are 40 ft (12.2 m) long. The drag factor for the truck is 0.70. How fast was the truck going right before and right after the pedestrian strike?

Solution

Since there was no deviation in the skid marks upon impact, the problem is constrained to an in-line (collinear) configuration. Also, the momentum of the pedestrian is very low since both the mass and speed are low as compared to the truck. This sets up the in-line momentum problem in which one object is initially at rest. So

$$w_1 v_1 = w_1 v_3 + w_2 v_4 \tag{12.22}$$

The post-impact speed of the truck is determined by a slide-to-stop equation:

US	SI
$v_3 = \sqrt{30df}$	$v_3 = \sqrt{254df}$
$= \sqrt{30(50)(0.70)}$	$= \sqrt{254(15.2)(0.70)}$
$= 32.40\,\text{mph}$	$= 51.98\,\text{kph}$

Now we can solve Eq. (12.22) for v_1 by dividing through by w_1:

$$v_1 = v_3 + \frac{w_2}{w_1}v_4 \tag{12.23}$$

So far we have said nothing about the post-impact speed of the pedestrian. However, we can see from Eq. (12.23) that if the speed of the truck changed by 1 mph (1.6 kph), then the second term of Eq. (12.23) must equal 1 mph (1.6 kph):*

US	SI
$1 = \frac{w_2}{w_1}v_4$	$1.6 = \frac{w_2}{w_1}v_4$

Rearrange, solving for v_4:

$v_4 = (1)\frac{w_1}{w_2}$	$v_4 = (1.6)\frac{w_1}{w_2}$
$= \frac{4500}{150} = 30\,\text{mph}$	$= (1.6)\frac{2041}{68} = 48\,\text{kph}$

Similarly, if the truck underwent a two-mile-per-hour change during the collision, the pedestrian would have experienced a 60-mph change in his or her speed. The physical evidence would be

*A 1-mph (1.6 kph) change means that $v_1 = 33.40$ mph (53.58 kph) and $v_3 = 32.40$ mph (51.98 kph).

significantly different for this case.

The point of this example is that when a moving object that has a lot of momentum due to its mass hits a small object, the speed change in the large object is insignificant. Also, the speed change of the lighter object is very sensitive to small speed changes of the heavier object.

When the one object weighs more than 10 times the other object, then speed changes of the smaller object become very sensitive. The larger object, however, is not affected much. The conservative approach is to ignore the speed change of the larger object. Let's continue with the example:

The exit velocity of the pedestrian was determined to be 30 mph (48 kph). Therefore, the entrance speed of the truck was 33.4 mph (53.57 kph). The speed at the beginning of the skid for the truck is determined using a combined speed formula:

US	SI
$S_o = \sqrt{30 d f + S_f^2}$	$S_o = \sqrt{254 d f + S_f^2}$
$= \sqrt{30(40)(0.70) + 33.4^2}$	$= \sqrt{254(12.2)(0.70) + 53.57^2}$
$= 44.22\,\text{mph}$	$= 70.98\,\text{kph}$

If the impact with the pedestrian is ignored, then:

$S_o = \sqrt{30 d f}$	$S_o = \sqrt{254 d f}$
$= \sqrt{30(40+50)(0.70)}$	$= \sqrt{254(12.2+15.2)(0.70)}$
$= 43.47\,\text{mph}$	$= 69.79\,\text{kph}$

This is less than one-mile-per-hour difference! This is often the case for collisions involving large mass ratios. For example, when a car hits a pedestrian or a dump truck hits a car.

12.6.6 Large Momentum Ratios

We distinguish large mass ratios from large momentum ratios because the two are not mutually inclusive. For example, if two vehicles have about the same mass, but one is going much faster than the other, then its momentum will be much greater than that of the other vehicle. In such situations, we find that we may say a lot about the speed of the vehicle with the larger momentum. However, the relative range of speeds calculated for the vehicle with the lesser impact momentum will be larger. For example, consider the situation in which a car traveling 70 mph (110 kph) impacts a much slower vehicle of equal mass pulling out from a stop sign, resulting in a two-dimensional collision. Regardless of the impact configuration, we will be able to determine the speed of the faster vehicle to a reasonable degree of accuracy. Because the departure angles for both vehicles will be largely in the pre-impact direction of the faster vehicle, the solution for the slower vehicle becomes sensitive to this small angle. For example, the speed calculated for the faster vehicle using a departure angle of ten degrees will be similar to the speed we would calculate for it if we assumed the slower vehicle were stopped and the collision was collinear. As an exercise, set up such a problem and solve it, looking at the two-dimensional case and then the collinear case with the slower vehicle stopped.

Example 12.6. Two vehicles collide in a T-bone configuration at an intersection. The approach of vehicle 1 is toward 0°. The approach of vehicle 2 is toward 90°. The vehicles stick together and move off as a single unit at a departure angle of 7°. $v_3 = 42$ mph (68 km/h). $v_4 = 42$ mph (68 km/h). $w_1 = 4200$ lb (1909 kg). $w_2 = 3750$ lb (1705 kg). Calculate v_1 and v_2.

Solution

Begin by calculating v_2 with Eq. (12.10):

$$v_2 = \frac{w_1 v_3 \sin\theta}{w_2 \sin\psi} + \frac{v_4 \sin\phi}{\sin\psi}$$

US	SI
$v_2 = \frac{4200(42)\sin 7}{3750\sin 90} + \frac{(42)\sin 7}{\sin 90}$	$v_2 = \frac{1909(68)\sin 7}{1705\sin 90} + \frac{(68)\sin 7}{\sin 90}$
$= \frac{4200(42)(0.121)}{3750(1)} + \frac{(35)(0.707)}{1}$	$= \frac{1909(68)(0.121)}{1705(1)} + \frac{(68)(0.121)}{1}$
$= 10.77\,\text{mph}$	$= 17.43\,\text{kph}$

Next, calculate v_1 using Eq. (12.16):

$$v_1 = v_3\cos\theta + \frac{w_2 v_4\cos\phi}{w_1} - \frac{w_2 v_2\cos\psi}{w_1}$$

US	SI
$v_1 = 42(0.992) + \frac{3750(42)(0.992)}{4200} - \frac{3750(10.77)(0)}{4200}$	$v_1 = 68(0.992) + \frac{1705(68)(0.992)}{1909} - \frac{1705(17.43)(0)}{1909}$
$v_1 = 41.66 + 37.20 - 0$	$v_1 = 67.45 + 60.24 - 0$
$v_1 = 78.86\,\text{mph}$	$v_1 = 127.69\,\text{kph}$

If we recalculate this problem as a collinear collision with the second vehicle stopped and having the same post-impact movement, we see the impact speed of the first vehicle rises to 79.50 mph (128.73 kph), and the speed of the second vehicle is zero.

12.7 Summary

Conservation of Linear Momentum (COLM) is a powerful tool for collision analysis as long as we are aware of its limitations. This principle is derived directly from Newton's Second and Third Laws of Motion and has a sound basis in fundamental mechanics. Collision analysis using COLM

goes beyond the calculation of impact speeds. We may also calculate the change in momentum vectors for each vehicle, keeping in mind these vectors must be equal and opposite. The change in momentum vectors give us the principal direction of force (PDOF). We may calculate or graphically determine the PDOF angles if we know the departure and approach vector directions, along with the magnitudes of the change in momentum vector and the post-impact vector. From this information, we may visualize the PDOF on each vehicle and use this to assist us in determining occupant seating positions.

A COLM analysis depends on our ability to determine pre- and post-impact directions, as well as post-impact speeds. Use the idea of the impact circle for the direction determination, keeping in mind that we are interested in the direction of each vehicle at the moment of impact as well as the directions that both vehicles are going immediately after the collision forces cease. The mathematical analysis involved in determining the post-impact speed may be more in-depth than the actual number crunching involved in solving the COLM impact speed equations.

There are limitations to a COLM analysis and the investigator needs to be aware of these. Large mass or momentum ratios may cause the analysis to give an unrealistic range of impact speeds for the lighter vehicle or the vehicle with the lesser pre-impact momentum. Significant external forces may sometimes be dealt with by means of an additional impulse term in the COLM equation, but the investigator needs to understand that quantifying this external force is not always possible or practical.

The variables we input into the COLM equation, such as pre- and post-impact directions, vehicle weights, and post-impact speeds are not constants. Rather, these are stochastic variables, which means that there is a range of possible values. Therefore, we must do our very best to accurately determine the values for these variables so as to make the range of each variable as small as possible. We will do a COLM analysis for our preferred set of variables first, and then we should range our answers using what we believe are the upper and lower limits of the variables. Keep in mind that the more likely solution is the first one as the chances of all lower values or all higher values stacking together are less. There are statistical (stochastic) methods to deal with this ranging problem, but these are beyond the scope of this discussion.

Exercises

Exercise 12.1. A Chevrolet Caprice weighing 3600 lb and a Ford Crown Victoria weighing 3800 lb are involved in a collision. The Chevrolet may be assumed to be on the x-axis with an approach

of 0°. The approach of the Ford is at 80°. The departure of the Chevy is 30° and the departure of the Ford is 45°. The departure speed of the Chevy is 30 mph and the departure speed of the Ford is 35 mph. Solve the problem both graphically and with equations.

$\boxed{39.58\,\text{mph}}$ a.) What is the speed of the Ford at impact?

$\boxed{44.42\,\text{mph}}$ b.) What is the speed of the Chevy at impact?

$\boxed{22.85\,\text{mph}}$ c.) What is the Δv of the Ford in the collision?

$\boxed{24.11\,\text{mph}}$ d.) What is the Δv of the Chevy in the collision?

$\boxed{61.46^\circ}$ e.) What is the α angle of the PDOF for the Ford?

$\boxed{38.47^\circ}$ f.) What is the α angle of the PDOF for the Chevy?

$\boxed{\text{Yes}}$ g.) Are the PDOF angles opposite in direction?

$\boxed{176,757\,\text{ft-lb}}$ h.) How much energy is lost in the collision phase of this accident?

Exercise 12.2. An oblique intersection is the scene of a crash in which an eastbound vehicle (#1) is struck by a vehicle coming from the northeast. The measured scene data gives $v_3 = 20$ mph, $v_4 = 35$ mph, $w_1 = 3000$ lb, $w_2 = 3500$ lb, $\theta = 315°$, $\phi = 260°$, and $\psi = 220°$. Determine the values of the following and verify your results using the graphical solution:

a.) v_1

b.) v_2

c.) Δv_1

d.) Δv_2

e.) α_1

f.) α_2

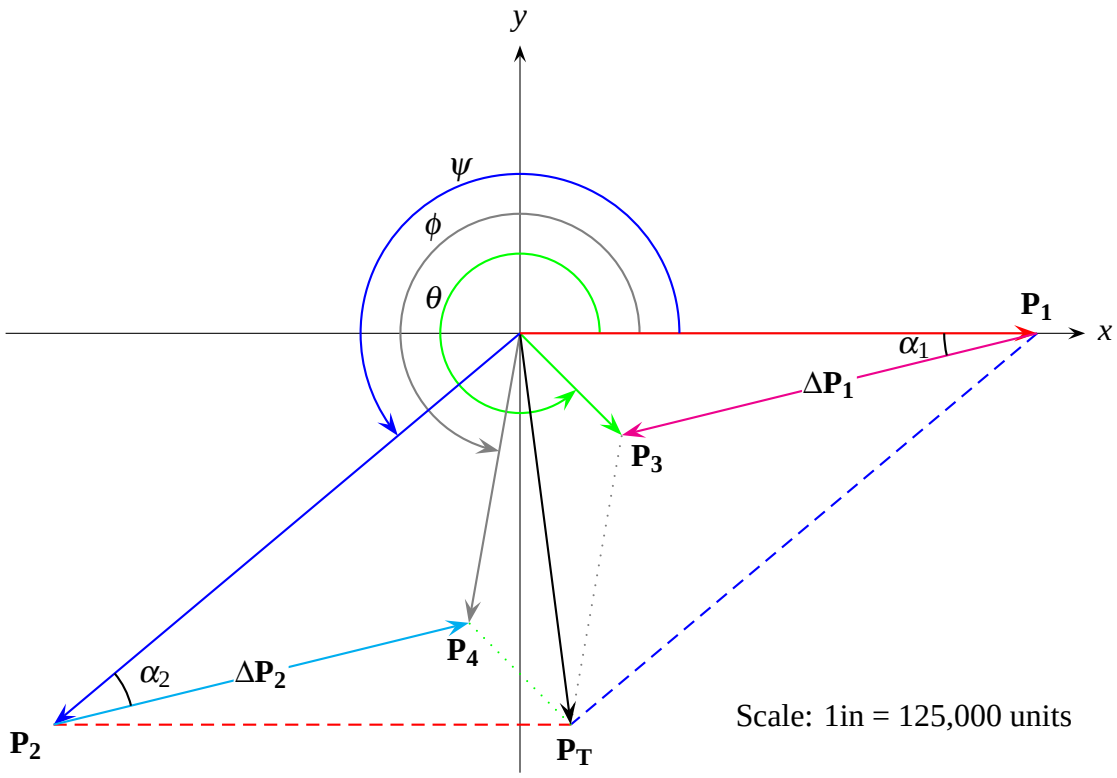


Figure 12.21: The graphical solution to Exercise 12.2.

Exercise 12.3. A country road intersection is the scene of a crash in which an eastbound vehicle (#1) is struck by a vehicle traveling 15° west of north.The measured scene data gives $v_3 = 55$ kph, $v_4 = 35$ kph, $w_1 = 2000$ kg, $w_2 = 1500$ kg, $\theta = 20°$, $\phi = 65°$, and $\psi = 105°$. Determine the values of the following and verify your results using the graphical solution:

a.) v_1

b.) v_2

c.) Δv_1

d.) Δv_2

e.) α_1

f.) α_2

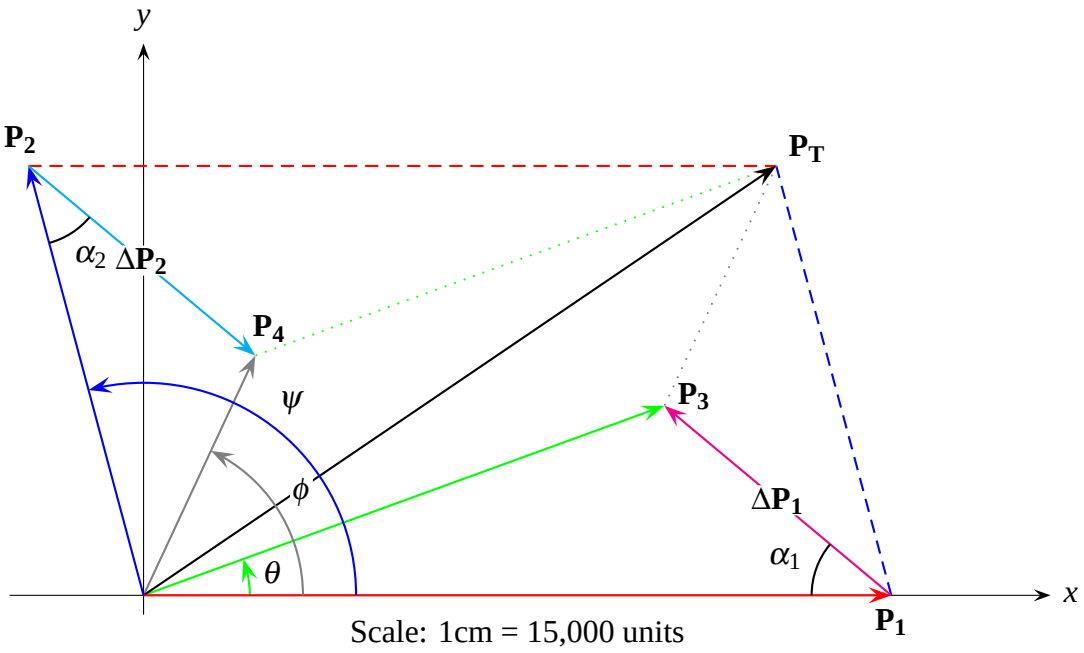


Figure 12.22: The graphical solution to Exercise 12.3.

13 Work and Energy

13.1 Work

In common language, work has come to mean exertion or effort, primarily in human terms. If we have had a long day at the office, we may say that we have worked hard. By this definition, writing this book was a lot of work! If we push hard on an immovable object and get physically tired, we say that this activity was hard work. However, none of these perceptions about work will help us very much to investigate traffic accidents.

In the study of mechanics, work has a very precise meaning:

Work is a force acting through a displacement. In mathematical form we use the dot product from Section 6.6.2 on page 6-11:

$$W = \mathbf{F} \cdot \mathbf{d} \tag{13.1}$$

where **F** is the force vector and **d** is the displacement vector. If the force and displacement vectors are parallel or constrained to one dimension, then the work is simply the scalar product of the magnitude of the force and distance. The unit of measure used to describe the amount of work done is the foot-pound (ft-lb) in the US system and the newton-meter or Joule (N-m or J) in the metric (SI) system. There are 4.45 newtons per pound and 0.3048 meters per foot. Thus, 1 Joule = 0.7376 ft-lb.

Example 13.1. How much work is done by pushing a refrigerator?

If we push a refrigerator away from the wall in order to clean under it, we have done some work because we have exerted a force through a displacement. If the refrigerator requires a 100-lb (445 N) force parallel to the floor to start it moving and to keep it moving, and we move it three feet (0.91 m), then we have done 300 ft-lb (406 Joules) of work. After looking at Figure 13.1 on page 13-2,

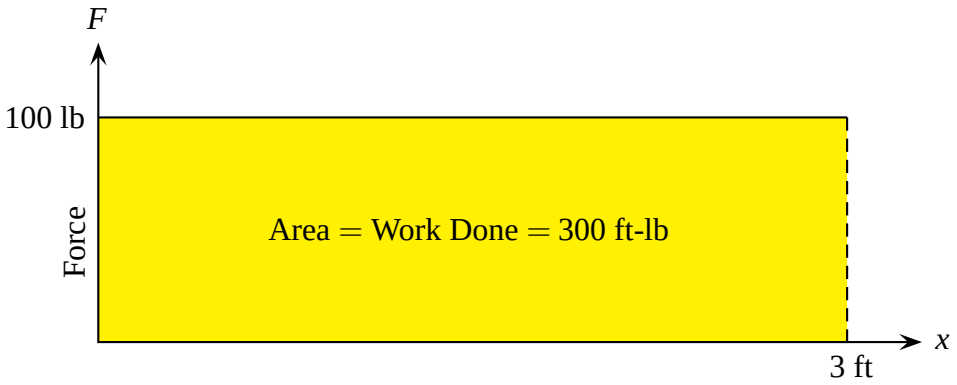


Figure 13.1: The work done by a constant force is the area of the rectangle described by the force and the distance.

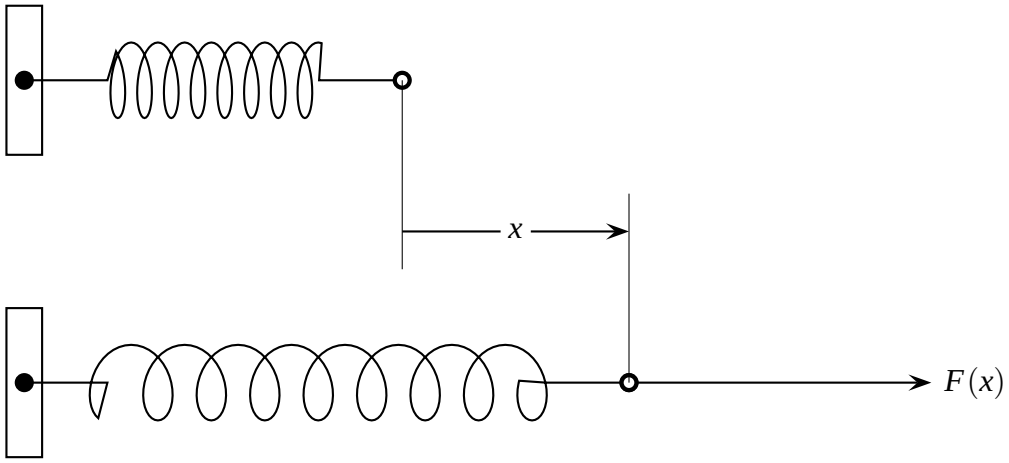


Figure 13.2: The force required to stretch a spring depends on how much the spring has already been stretched.

we see the graph of the force vs. displacement curve is a rectangle. Since work is the product of force and displacement, it is the product of one side of the rectangle multiplied by the adjacent side. If we multiply one side of a rectangle by an adjacent side, we get the area of the rectangle. Therefore, the work done is the area under the force-displacement curve in Figure 13.1.

In Example 13.1, we have assumed a constant force. However, a force acting through a displacement need not be constant. Let us look at another example, that of an ideal spring. As we stretch a spring, the force required to increase the displacement of the spring steadily increases.

This is illustrated in Figure 13.2 and can be described by the relationship known as *Hooke's Law*:

$$F = kx \tag{13.2}$$

where F is force in pounds or newtons,

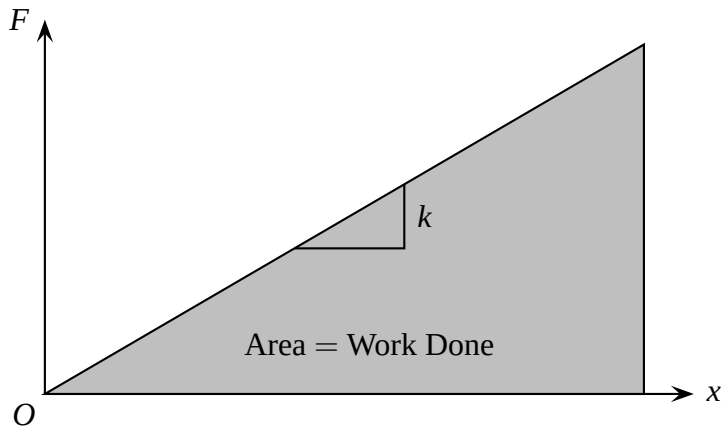


Figure 13.3: Force deflection response of an ideal spring where k is the slope of the line. The area inside the triangle represents the work done to deflect the spring.

k is the spring constant in lb/in or N/cm, and
x is the displacement in inches or cm.

Note: x can be used interchangeably with d to denote displacement.

We see that the area under the force–displacement curve shown in the graph of Figure 13.3 is a right triangle. From elementary geometry, we may write the following equation for the work required to stretch a spring (which is simply the equation that describes the area of a triangle):

$$Area = \frac{1}{2}bh$$

where b is the base of the triangle and h is the height of the triangle. In this example, $b = x$ (the displacement), h is defined as kx (the force), and $Area = W$ (work). Thus:

$$W = \frac{1}{2}kx^2$$

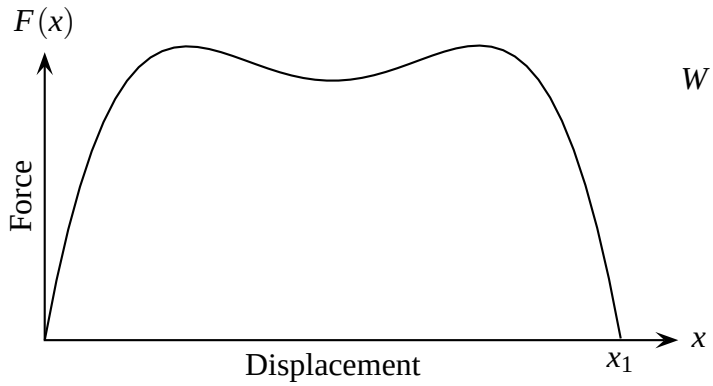


Figure 13.4: An irregular force–displacement curve in the shape of a typical curve due to friction.

We may also use calculus to derive the same equation:

$$\begin{aligned} W &= \int_o^x F(s)\,ds \\ &= \int_o^x ks\,ds \\ &= \frac{1}{2}k\left(s^2\big|_0^x\right. \\ &= \frac{1}{2}k(x^2-0^2) = \frac{1}{2}kx^2 \end{aligned}$$

Now consider the case of an irregular force acting over displacement, as illustrated in Figure 13.4. The total work done by this force is the area under the curve, and is represented by the integral of the force curve as a function of displacement.

In order for us to mathematically integrate this irregular curve, we must know something about the curve. Perhaps a portion of the curve can be modeled as a known curve, while another portion is perhaps a line segment with a known slope. We could calculate the area under the curve by using these discrete elements, assuming they could be identified.

However, there is a simpler way to do this. Let us break the curve into very small discrete rectangles like the one shown in Figure 13.5. The value of the force at the top of each rectangle is very nearly constant for the narrow width of the rectangle. We may then calculate the area of each rectangle and add them together. Doing this will give us a close approximation to the true area under the curve and hence a very close approximation to the work actually done. This is a very old technique and was used even before the invention of calculus. Refinements of this technique are known as

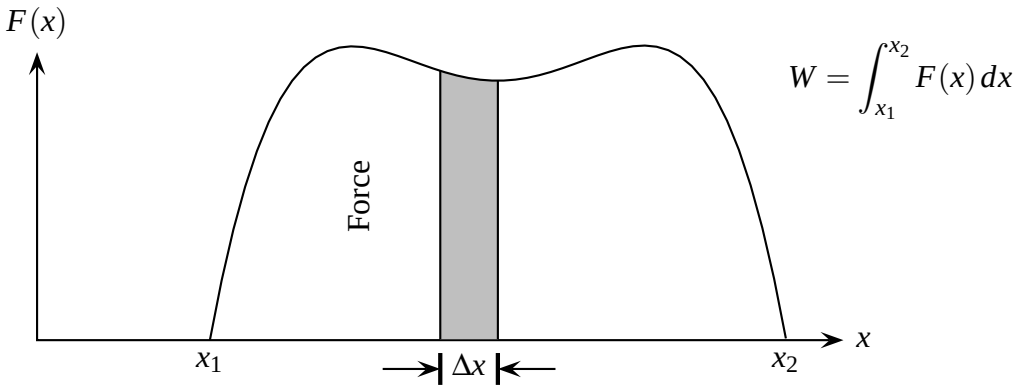


Figure 13.5: A process of numerical integration involves taking small sections of the curve and approximating them as rectangles.

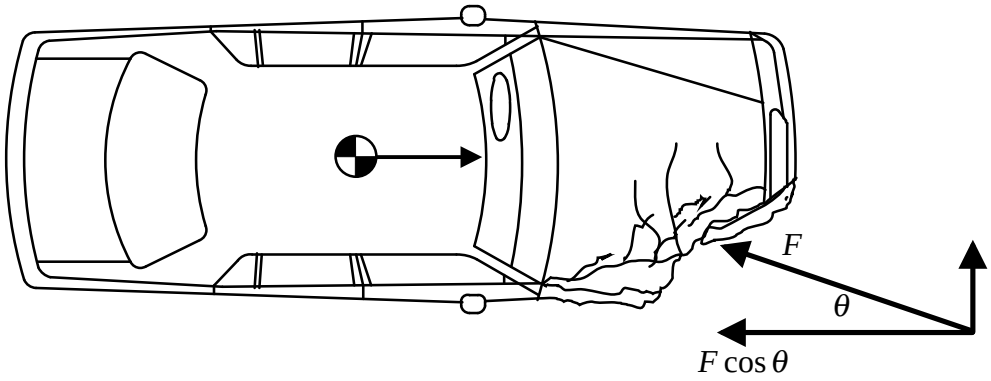


Figure 13.6: Force acting to crush a car can be broken into components to determine the normal force and the tangent force.

numerical integration (see Section **??** on page ??). The force of friction acting on a stopping vehicle is similar in form to the curve in Figure 13.4. Consequently, as we continue our study of traffic crash investigation and reconstruction, we will see curves of similar form from accelerometer data taken from stopping vehicles.

In our discussion thus far, we have assumed that the force and the displacement are both in the same direction. This is not always the case. Consider for a moment a car that is involved in a collision. The collision force **F** is acting at some angle with respect to the car, as illustrated in Figure 13.6. Only part of this force does work to crush the car straight back. If we know the magnitude of the collision force, then we can resolve it into components parallel to the car and tangent to the car. The parallel component of the collision force is the force that actually crushes the car.

This parallel component is $F\cos\theta$ in Figure 13.6, and the work done to crush the car would be

$W = x(F\cos\theta)$. We can see that the work done is the component of force in the direction of the displacement times the magnitude of the displacement x. In other words, work is the product of the force times the displacement times the cosine of the angle between the two. This brings up the idea of a workless force. If a force is acting at right angles to the motion of the object, then that force does no work.

In our discussion of Figure 13.6, we have just restated the definition of the dot product of two vectors! (see Section 6.6.2 on page 6-11). Thus, work, although the product of two vectors (force and displacement), is actually a scalar quantity. Since work is a scalar quantity, there is no direction associated with it. It takes just as much work to replace the refrigerator as it does to move it out from the wall in the first place. In the field of traffic crash investigation and reconstruction, the CRASH III algorithm uses the dot product idea to develop a model, based upon a simple force-displacement curve and empirically derived crush coefficients, that will calculate the work required to crush a vehicle. When we use the CRASH III model, we measure the crush perpendicular to the front or to the side of the vehicle, depending on the surface involved in the collision.

13.2 Mechanical Energy

13.2.1 Kinetic Energy

Kinetic energy is possessed by any body in motion. Translational kinetic energy is energy that relates to the speed of an object moving along a path. Rotational kinetic energy is energy possessed by a body due to rotation around any point. Rotational kinetic energy usually adds very little to the total energy possessed by a moving vehicle, even in the case of a spin. We should be aware that rotational kinetic energy exists, but we will reserve discussion to Section 14.7 on page 14-19.

The equation for translational kinetic energy is:

$$KE = \frac{1}{2}mv^2 \tag{13.3}$$

where m is the mass and v is the magnitude of the velocity vector. The derivation of this equation is in Section 13.5 on page 13-15.

Equation 13.3 is a general form of the kinetic energy equation and should be committed to memory. However, when dealing with common units such as pounds and miles per hour in the US system or

kilograms and kilometers per hour in the SI system, another form of the kinetic energy equation is useful.

US System of Measurement

Let us begin by converting speed in miles per hour to velocity in feet per second:

$$v = \frac{5280\,\text{feet/mile}}{3600\,\text{seconds/hour}} S = 1.466S$$

where S is speed in miles per hour and v is velocity in feet per second.* Now substitute this value for v into Eq. (13.3):

$$KE = \frac{1}{2} m (1.466S)^2$$

Now, recall from Section 10.3 on page 10-3 that $m = \frac{w}{g}$ so:

$$\begin{aligned} KE &= \frac{1}{2}\left(\frac{w}{g}\right)(1.466S)^2 \\ &= \frac{w}{2(32.2)}(1.466S)^2 \\ &= \frac{2.14wS^2}{64.4} \end{aligned}$$

Performing the arithmetic gives a more convenient formula for kinetic energy in the US system:

$$\boxed{KE = \frac{wS^2}{30}} \tag{13.4}$$

Rearranging this equation enables us to solve for a speed:

$$\boxed{S = \sqrt{\frac{30KE}{w}}} \tag{13.5}$$

where S is the speed of an object in miles per hour, w is the weight of the object in pounds, and KE is the kinetic energy in ft-lb.

*Actually, the magnitude of velocity and speed are exactly the same thing. The S and v notation is used as a reminder of the units. A bold face **v** will be used to denote velocity as a vector quantity.

SI System of Measurement

Begin by converting speed in kilometers per hour to velocity in meters per second:

$$v = \frac{1000\,\text{meters/kilometer}}{3600\,\text{seconds/hour}} S = \frac{S}{3.6}$$

where S is speed in kilometers per hour and v is velocity in meters per second. Now substitute this value for v into Eq. (13.3):

$$\begin{aligned} KE &= \frac{1}{2} m \left(\frac{S}{3.6} \right)^2 \\ &= \frac{mS^2}{2(12.96)} \end{aligned}$$

Performing the arithmetic gives a more convenient formula for kinetic energy in the SI system:

$$\boxed{KE = \frac{mS^2}{26}} \tag{13.6}$$

where S is the speed of an object in kilometers per hour and m is the mass of the object in kilograms. The kinetic energy, KE, will have units of Joules.

Rearranging Eq. (13.6) enables us to solve for a speed:

$$\boxed{S = \sqrt{\frac{26KE}{m}}} \tag{13.7}$$

Kinetic energy is the energy possessed by an object due to its motion. Kinetic energy also represents the amount of work that an object in motion can do. The concept that energy can do work is expressed in the Work–Energy Theorem and is presented in Section 13.4 on page 13-14. Before we talk about the Work–Energy Theorem, let's look at some other forms of energy and some laws governing energy.

13.2.2 Potential Energy

Potential energy is stored energy. Potential energy is the energy an object has based on its position with respect to a reference level. One of the simplest way to store energy is to place an object

high off the ground. The object will possess potential energy due to its position above the ground.[†] This is the principle of storing energy with water. A dam will keep the water at bay while it gains altitude. As the water falls, it can do work by pushing on a turbine. The potential energy stored by an object due to its height has the following equation:

$$PE = mgh \tag{13.8}$$

where m is the mass of the object,
g is the acceleration due to gravity, and
h is the height of the object.

If we look closely at Eq. (13.8), we can see that $mg = w$ and this weight is a force. Obviously h is a distance, so potential energy is a product of force and distance, which we know is work. This means we can think of potential energy as the amount of work required to move a force through some distance. We can also express potential energy due to height as

$$PE = wh \tag{13.9}$$

In another example, let's consider a stretched spring. Stretching a spring requires some work. If a spring is able to return to its original shape (i.e., the spring is elastic), then the spring can do the same amount of work it took to initially deform the spring. Since we stored this work by moving a variable force though a distance, we can express the potential energy of a spring as:

$$PE = \frac{1}{2}kx^2 \tag{13.10}$$

If all of the energy used in deforming the spring is returned, the work done to deform the spring is said to be *reversible.* If the work done to deform the spring is not returned and the deformation is permanent, then the work done to deform the spring is *irreversible*. A permanently deformed spring has no potential energy; however, if the amount of deformation is known, then the amount of energy absorbed is the same amount of work done in permanently damaging the spring, which is given by Eq. (13.10).

Energy from height and energy stored in a spring (elastic) are the most common forms of potential energy used in reconstruction. Other forms include chemical energy, heat energy, sound energy, and electromagnetic energy. These forms of energy are important because they follow the law of

[†]If the object were on the ground, then it would possess no potential energy with respect to the level of the ground.

the conservation of energy.

Mechanical energy is the sum total of all the potential and kinetic energy of a system.

13.3 Conservation of Energy

The law of the conservation of energy is also known as the First Law of Thermodynamics. It can be simply stated as:

> Energy cannot be created or destroyed. It is only transferred from one form to another.

This has very practical implications to traffic crash reconstruction because the energy dissipated in a skid and/or crash had to come from somewhere. The primary source for this energy is the kinetic energy the vehicle possesses at the beginning of the maneuver. So why are we allowed to say that the dissipated energy comes from kinetic energy? The reason is because we can treat the vehicle involved in the crash as a closed system, or sometimes an isolated system.

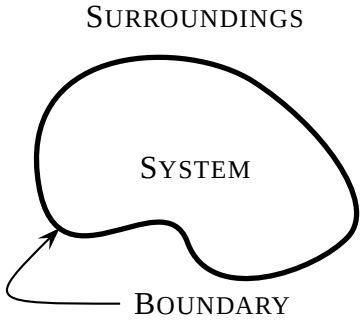


Figure 13.7: A system is a region that has a boundary and surroundings.

13.3.1 Systems

A system is defined as a quantity of matter or a region of space under consideration. This is a really broad term, but it is necessary to describe the way energy behaves. Figure 13.7 shows a general system with a boundary and its surroundings. A vehicle can be considered a system with the surface of the road as a boundary and the rest of the earth as its surroundings.

A closed system is a system in which no mass is transferred across the boundary. In general, we can treat a vehicle as a closed system. A counterexample to this would be a dump truck losing its load. If the dump truck is the system, then its mass is changing as the load is being removed from the system. Energy, however, can be transferred across the boundary of a closed system.

An isolated system is a system in which the total quantity of mass and energy is fixed within the system. Therefore, we can draw a boundary around all the objects and interaction that contain and transfer energy. This boundary contains a fixed amount of energy. In reality, there is no such thing as a purely isolated system because there is always some interaction with the surroundings. However, the magnitudes of those interactions are often so small that they do not need to be considered. For example, a car falling off a bridge is an isolated system. The total amount of energy remains the same as the potential energy is turned into kinetic energy. The influence of the air (surroundings) is negligible.

13.3.2 Conservative Forces

An example of a conservative force is gravitational force. This means the change in potential energy when going from one point to another is independent of the path the object takes in changing heights. If someone were to throw a ball off a second floor, its change of potential energy from the second floor to the first floor is a fixed amount, regardless if the ball is dropped, thrown out, tossed straight up in the air, or rolled down a ramp.

Let's look how we can use the law of the conservation of energy to determine the velocity of an object under the influence of gravity. We will assume that the object has some initial kinetic energy expressed as:

$$KE_o = \frac{1}{2}mv_o^2 \tag{13.11}$$

We will also define the starting point to have zero height so that a gain in potential energy indicates the object's movement further from the earth. Likewise, a loss in potential energy is represented as a negative quantity, which indicates that the object is getting closer to the earth.

The kinetic energy of an object at any time is the initial kinetic energy plus the loss in potential energy. The potential energy is turned into kinetic energy as an object falls.

$$KE = KE_o + mgh \tag{13.12}$$

where h is the height the object has fallen. If we expand Eq. (13.12) by substituting in the formula for kinetic energy, we get:

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_o^2 + mgh \tag{13.13}$$

Notice that each term in Eq. (13.13) has mass, so we must multiply through by $\frac{2}{m}$:

$$v^2 = v_o^2 + 2gh \tag{13.14}$$

Now take the square root of both sides:

$$v = \sqrt{v_o^2 + 2gh} \tag{13.15}$$

In this case, g is the acceleration due to gravity and h is the height fallen. Since gravity is a conservative force, the vertical velocity at any height h is the same no matter how it got there. Therefore, if we throw a ball straight up, it will have an initial velocity v_o. The ball will go up and come back down. When it falls back to the same level as the takeoff, $h = 0$, the potential energy is zero. Thus, all the energy of the ball came from its initial kinetic energy, and the ball will be going the same speed as when it was thrown.

We can make Eq. (13.15) more general by substituting any acceleration a for g and any distance d for h. Then Eq. (13.15) becomes:

$$v = \sqrt{v_o^2 + 2ad}$$

Which is the same as Eq. (7.14) on page 7-14, which we derived by using kinematic principles. However, if we use any acceleration, or distance, the forces that create the acceleration may not be conservative.

13.3.3 Non-Conservative Forces

Friction is the primary non-conservative force we deal with in physics. Aerodynamic drag is even a form of friction. When an object moves while a frictional force is acting upon it, the energy being used to perform the work is being dissipated as some other form of energy. These other forms may be heat energy, electric, potential, or even sound.

The work done by non-conservative forces is irreversible. None of the mechanical energy is stored,

so the total amount of energy a system possesses is always decreasing when moving under the influence of a non-conservative force such as friction.

Work done by a non-conservative force depends on the path taken from start to end. If we use the definition of work based on calculus, then the integral would be a path integral. Therefore, the equation defining work based on an average non-conservative force is simply:

$$W = Fd$$

where d is the actual distance traveled.

Example 13.2. How much work is done by moving a box across a table when the coefficient of friction is 0.5 if:

1. the box is moved in a straight line?
2. the box is moved in a circular arc?

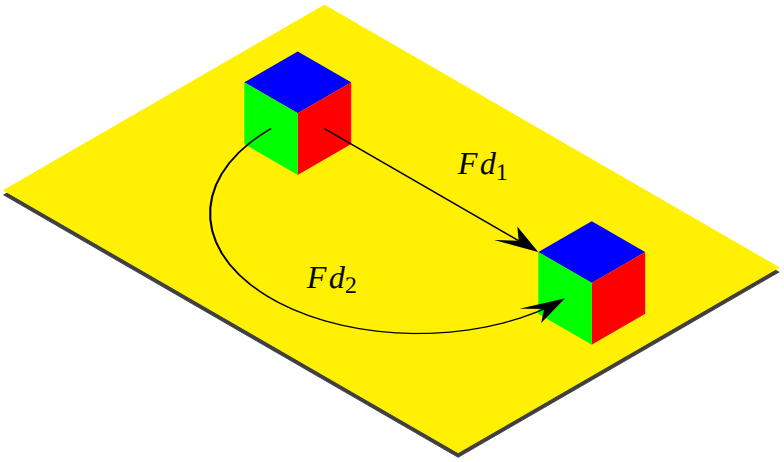


Figure 13.8: The path of the box is either straight or in a circular arc. The amount of work done by each movement is different even though the box ends at the same place.

Solution

We begin by drawing a small sketch, as shown in Figure 13.8. We see that the distance d_2 is half the circumference of a circle with a diameter of d_1. This means that $d_2 = \pi d_1/2$. The force needed to push the box is always opposite the direction of travel and is equal to the coefficient of friction

times the weight, w. So the work done in moving the box along d_1 is :

$$W_1 = \mu w d_1$$

and the work done by moving the box along the circular path is

$$W_2 = \frac{\pi \mu w d_1}{2}$$

As we can see, the work done in the circular path is $\frac{\pi}{2}$ times greater than the work done in the straight line. This calculates out to be 1.57 times as much. As a result, when considering non-conservative forces, such as friction, the path of the force is important.

13.4 Work–Energy Theorem

The Work–Energy Theorem is the fundamental building block for deriving many of the equations we use in traffic crash reconstruction. It can succinctly be stated as:

> The work done by a force on an object is equal to the change in kinetic energy of the object.

This theorem is written mathematically as:

$$W = \Delta KE = KE_f - KE_o$$

where W is the work done to the body,
KE_f is the kinetic energy of the body after the work was done, and
KE_o is the kinetic energy of the body before the work was done.

By substituting in the appropriate expressions for work and kinetic energy we arrive at:

$$\begin{aligned} Fd &= \frac{1}{2}mv_f^2 - \frac{1}{2}mv_o^2 \\ &= \frac{1}{2}m(v_f^2 - v_o^2) \end{aligned}$$

We must realize that the work done is a function of the change in kinetic energy, not just a change in velocity. A change in velocity, Δv, is simply $v_f - v_o$. Therefore, we cannot say that $Fd = \frac{1}{2}m(\Delta v)^2$

because $(v_f - v_o)^2 \neq (v_f^2 - v_o^2)$. This is a common misunderstanding and is fundamentally important to the derivation of the combined speed equations. IMPORTANT!

13.5 Derivation of the Kinetic Energy Formula

Let's think how we might develop a formula to express kinetic energy. We should remember Newton's Second Law, which we expressed as:

$$F = ma$$

Also, acceleration, a, is velocity, v, divided by time, t.

$$a = \frac{v}{t}$$

Work is defined as

$$W = Fd$$

where W is work, F is force, and d is the distance through which the force acts. By the Work–Energy Theorem, the work has the same magnitude as the kinetic energy of the body, which means $W = KE$.

So where does all of this lead us? Let's start with Newton's Second Law and substitute $\frac{v}{t}$ for a in this equation. This gives:

$$F = m\frac{v}{t}$$

At this point, we have digressed just a bit. We know $W = Fd$ but we need to relate d to something that we know, namely acceleration a. From Table 7.1 on page 7-15 we see:

$$d = vt$$

where v is the average velocity. It will also be important for us to keep track of which velocity is represented by v in the equations to follow: average velocity or final velocity. Let's consider a body that starts from rest: The body accelerates at a constant rate. We must remember that acceleration

is the time rate of change of velocity:

$$v = at$$

where v is the final velocity. How much distance does the body travel in a given time? To find the distance, we must find the average velocity. The equation looks like this:

$$v_{avg} = \frac{v_o + v_f}{2}$$

where v_{avg} is the average velocity, v_o is the initial velocity, and v_f is the final velocity. Since the initial velocity is zero, with the body starting from rest, the equation can be simplified to:

$$v_{avg} = \frac{v_f}{2}$$

Now remember that the body has been accelerating at rate a for time t to give a final velocity v_f. We may substitute at for v_f in the above equation:

$$v_{avg} = \frac{at}{2}$$

Recall that:

$$d = vt$$

where v in this case is the same as v_{avg}. Substitute $\frac{at}{2}$ for v or v_{avg} :

$$d = \frac{at}{2}t \qquad \text{or} \qquad d = \frac{1}{2}at^2$$

As we saw in the beginning of this section, $a = \frac{v}{t}$, where v is the final velocity v_f. Substitute $\frac{v}{t}$ for a in our last equation above:

$$d = \frac{1}{2}at^2 = \frac{1}{2}\left(\frac{v}{t}\right)t^2 \qquad \text{where } v = v_f$$

or

$$d = \frac{1}{2}vt$$

Now recall that kinetic energy equals work calculated by multiplying force and distance:

$$W = KE = Fd$$

Also recall that:

$$F = m\frac{v}{t}$$

Therefore, in the equation $KE = Fd$ we may substitute $m\frac{v}{t}$ for F and $\frac{1}{2}vt$ for d:

$$KE = m\frac{v}{t}\left(\frac{1}{2}vt\right)$$

This equation simplifies to:

$$KE = \frac{1}{2}mv^2 \tag{13.16}$$

This gives us the equation we are looking for, namely, an equation that demonstrates how much kinetic energy a body of a given mass moving at a given velocity possesses.

Now that we can quantify kinetic energy, what good will it do us? We will recall that it takes a certain amount of work to stop a moving vehicle. If we can determine the work required to stop the vehicle and if we know the mass of the vehicle, then we can determine the speed at which the vehicle had been moving. These concepts will be tied together in Chapter **??**, when we deal with speed equations.

13.6 Power

In physics, the word *power* has a very specific meaning that differs from that conveyed in our everyday use of the word. Because of his strength, we commonly speak of an NFL lineman as a person having tremendous power. We also use this word to describe the influence of someone who has authority or social position. However, in the study of physics, power means the time rate of doing work, and the focus is on the rate at which work is done. In mathematical terms:

$$P = \frac{W}{\Delta t} \tag{13.17}$$

where P is power,
Δt is the change in time, and
W is the work done.

We also know that work is force times distance

$$W = Fd \tag{13.18}$$

and that velocity is distance divided by time ($v = \frac{d}{\Delta t}$), which means that

$$\Delta t = \frac{d}{v} \tag{13.19}$$

Substitute Eqs. (13.18) and (13.19) into Eq. (13.17):

$$P = \frac{F \not{d}}{\frac{\not{d}}{v}}$$
$$P = Fv$$

Power is the velocity of the force[‡]. The most common unit for power in the US system is the horsepower (hp); the Watt is the most common unit in the SI system (see Section 5.2.2.9 on page 5-15).

As a vehicle stops, work is being done. If the vehicle hits a tree, it will stop much more quickly than if it slides on its tires from the same speed. The work done is the same, but the collision with the tree is a much more powerful event.

Example 13.3. A vehicle going 62 mph (100 km/h) strikes a tree and stops in 0.10 seconds. This vehicle weighs 3000 lb (1360 kg). How much power (average power over 0.10 seconds) does this develop?

Solution

This can be accomplished by utilizing Eq. (13.17), but we first must determine the amount of work the vehicle will do:

[‡]Only the force component in the same direction as the motion is doing work.

US	SI
$W = KE = \frac{wS^2}{30}$	$W = KE = \frac{mS^2}{26}$
$= \frac{3000(62)^2}{30}$	$= \frac{1360(100)^2}{26}$
$= 384,400\,\text{ft-lb}$	$= 523,076\,\text{J}$

Now we can calculate the power:

US	SI
$P = \frac{W}{\Delta t}$	$P = \frac{W}{\Delta t}$
$= \frac{384,400\,\text{ft-lb}}{0.10\,\text{sec}}$	$= \frac{523,076\,\text{J}}{0.10\,\text{sec}}$
$= 3,844,000\,\text{ft-lb/sec}$	$= 5,230,760\,\text{W}$

If we convert this to horsepower, 1 hp = 550 ft-lb/sec, so

$$P = \frac{3,844,000\,\text{ft-lb/sec}}{550\,\frac{\text{ft-lb/sec}}{\text{hp}}} = 6989\,\text{hp}$$

Example 13.4. An 80,000-lb (36,280-kg) tractor-trailer has an overall drag factor resistance of 0.02. How much power will it require to travel at 62 mph (100 km/h)?

Solution:

$$P = Fv$$

where P is power, F is force, and v is velocity in ft/sec (m/sec). We can calculate the resistive force due to the drag factor using the relationship $F = wf$.

US	SI
Recall that $v = 1.466S$ so	Recall that $v = \frac{S}{3.6}$ so
$P = 1.466wfS$	$P = \frac{mgfS}{3.6}$
Also, 1 hp = 550 ft-lb/sec so:	Also, 1 kW = 1000 watts so:
$P = \frac{1.466wfS}{550}$	$P = \frac{9.81mfS}{3.6(1000)}$
$P = \frac{wfS}{375}$	$P = \frac{mfS}{367}$
Substitute in appropriate values:	Substitute in appropriate values:
$P = \frac{80,000(0.02)(62)}{375}$	$P = \frac{36,280(0.02)(100)}{367}$
$P = 264.5\,\text{hp}$	$P = 197\,\text{kW}$

Example 13.5. What is the average power required to accelerate a 3500-lb (1587-kg) vehicle on a level road from 0 to 62 mph (100 km/h) in 5.0 seconds?

Solution

$$P = \frac{W}{\Delta t}$$

Recall KE_{gained} = work done from the Work–Energy Theorem. So let $W = KE$:

US

$$P = \frac{KE}{t}$$
$$KE = \frac{wS^2}{30}$$

Make the substitution:

$$P = \frac{wS^2}{30t}$$

Substitute in the conversion to horsepower along with other appropriate values:

$$P = \frac{3500(62^2)}{30(550)(5)}$$
$$P = 163\,\text{hp}$$

at the drive wheels.

SI

$$P = \frac{KE}{t}$$
$$KE = \frac{mS^2}{26}$$

Make the substitution:

$$P = \frac{mS^2}{26t}$$

Substitute in the conversion to kilowatts along with other appropriate values:

$$P = \frac{1587(100^2)}{26(1000)(5)}$$
$$P = 122\,\text{kW}$$

at the drive wheels.

This problem assumes that the force generated at the drive wheels is the only force present. In reality, there is an aerodynamic effect that requires more power to accelerate with this same type of performance. This example actually determined the lowest possible power required when we neglect wind resistance.

Example 13.6. An 80,000-lb (36,280-kg) tractor-trailer can develop 350 hp (261 kW) at the drive wheels. Neglecting shift times, what is the minimum amount of time required to accelerate to 50 mph (80.46 km/h)?

Solution

The equations from Example 13.5, along with the correct conversion factor to get power in horsepower (kilowatts), give the following relationship for power:

US	SI
$P = \frac{wS^2}{16,500t}$	$P = \frac{mS^2}{26t}$
Solve for t:	Solve for t:
$t = \frac{wS^2}{16,500P}$	$t = \frac{mS^2}{26,000P}$
$= \frac{80,000(50^2)}{16,500(350)}$	$= \frac{36,280(80.46^2)}{26,000(261)}$
$= 34.63\,\text{seconds}$	$= 34.61\,\text{seconds}$

Again, this example does not take into account the increasing wind resistance on the truck as it accelerates.

Exercises

Exercise 13.1. An object is pulled 35 ft across a horizontal surface with a force of 300 lb. How much work was done by the force? $\boxed{10,500\,\text{ft-lb}}$

Exercise 13.2. An object is pulled 35 ft across a horizontal surface with a force of 300 lb. The 300-lb force makes a 25 degree angle to level. How much work was done by the force? $\boxed{9516\,\text{ft-lb}}$

Exercise 13.3. A box is pulled 45 ft up a 35° incline with a force of 120 lb. The force is parallel to the surface. How much work was done by the force? $\boxed{5400\,\text{ft-lb}}$

Exercise 13.4. An object is raised with a constant velocity 50 ft above a horizontal surface with a force of 1300 lb. How much work was done by the force? $\boxed{65,000\,\text{ft-lb}}$

Exercise 13.5. What is the weight of the object in the above question? $\boxed{1300\,\text{lb}}$

Exercise 13.6. An external unbalanced force of 440 newtons is applied to a box of 100 kg and moves it 5 meters.

a.) How much work was done by the force? $\boxed{2200\,\text{J}}$

b.) How much kinetic energy is gained by the box? $\boxed{2200\,\text{J}}$

c.) If the box was initially at rest, how fast is it moving? $\boxed{6.63\,\text{m/s}}$

d.) If the box has an initial velocity of 1 m/s, what is its initial kinetic energy? $\boxed{50\,\text{J}}$

e.) If the box has an initial velocity of 1 m/s, what is its final velocity after having the above work done to it? $\boxed{6.7\,\text{m/s}}$

Exercise 13.7. A spring with a mass of 1 kg hanging lets the mass sit at the 10 cm mark. When 2 kg are added to the mass to make a total of 3 kg, the mass sits at 20 cm. What is the spring constant (slope) in N/cm? $\boxed{1.96\,\text{N/cm}}$

Exercise 13.8. A 2100-kg vehicle takes off up a 30 degree ramp at 72 kph and falls to a landing point that is 40 meters below its takeoff point.

314.0 kJ a.) What is the kinetic energy of the vehicle when it reaches the peak height of its trajectory?

927.9 kJ b.) What is the potential energy of the vehicle when it reaches the peak height of its trajectory?

418.7 kJ c.) What is the kinetic energy of the vehicle at takeoff?

1241.9 kJ d.) What is the kinetic energy of the vehicle at landing?

1241.9 kJ e.) What is the total mechanical energy of the vehicle at any point along its trajectory in relation to a landing point that is 40 meters below its takeoff point?

Exercise 13.9. A 3800-lb Ford Crown Victoria is attached to a winch system that is capable of pulling with a continuous 3500-lb force. This winch is used to accelerate the Ford from a stop over a distance of 25 ft.

26.28 mph a.) If the Ford rolls without friction, what is its speed in mph at the end of the 25 ft?

11.32 mph b.) If the brakes of the Ford are locked and the friction coefficient is 0.75, how fast will the Ford be going in mph at the end of the 25 ft?

87,500 ft-lb c.) How much work is done by the winch in part a?

87,500 ft-lb d.) How much work is done by the winch in part b?

87,500 ft-lb e.) What is the kinetic energy of the Ford in part a?

16,250 ft-lb f.) What is the kinetic energy of the Ford in part b?

14 Rotational Mechanics

14.1 Uniform Circular Motion

This section involves motion in two dimensions because a circle is two dimensional. The various quantities of motion are described by vectors. The concepts of motion in more than one dimension from Chapter 8 should be understood.

Uniform circular motion is nothing more than the movement of a body in a circular (constant radius) path at a constant speed. The acceleration vector is always pointing toward the center of the circle and the velocity vector is tangent to the circle.

In traffic crash investigation, uniform circular motion is important to us when a vehicle leaves the road in a critical speed yaw situation.

Before proceeding, we will have to consider the difference between velocity and speed. As we will recall from an earlier section, velocity is specified by both a magnitude and a direction, whereas speed is specified by only a magnitude. Reason tells us that the velocity of a body moving in a circular path at a constant speed is constantly changing, since the direction of the body is constantly changing. We will recall that a change in velocity results in an acceleration. Furthermore, remember that a mass undergoing a change in velocity has a force acting upon it. Thus, we may conclude that an external unbalanced force is acting on a body to cause it to move in a circular path. In the case of a ball twirling at the end of a string, the string provides the force to make the ball move in a circular path. In the case of a vehicle on a road surface, the force of friction between the tires and the road keeps the vehicle moving in a circular path. The external unbalanced force may have various sources, but tire forces are the primary sources for vehicles.

We now need to see if there is some way to express uniform circular motion mathematically. To do this, we need to make a diagram like Figure 14.1, in which the body is traveling with a constant speed in a circular path with a radius of r. At point A, the body has a velocity in a direction to the right. A short time later, at point B, the body has a velocity of the same magnitude as the first

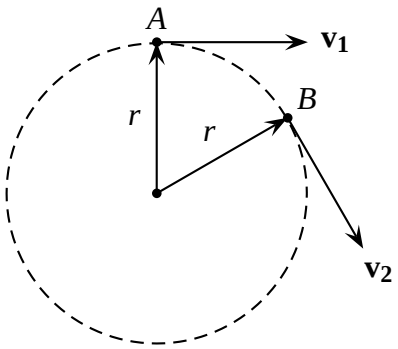


Figure 14.1: An object traveling in a circular path at a constant speed.

velocity, but in a different direction. We will call these velocities $\mathbf{v_1}$ and $\mathbf{v_2}$. What we are really interested in is the change occurring from $\mathbf{v_1}$ to $\mathbf{v_2}$ since it is the change in velocity with respect to time that results in an acceleration. We may write this change in velocity as $\mathbf{v_1} - \mathbf{v_2}$. Let's redraw the velocities in Figure 14.1 from a common point, and then connect the ends of the velocities with another vector. This vector will represent the change in velocity, and is the product of the average acceleration vector and time, $\mathbf{a}t$. The magnitude of the acceleration vector is written as a, and t is the time interval required to go from A to B. The results of this are shown in Figure 14.2.

We may also draw a third diagram as Figure 14.3, which will relate to the circular path that is being followed by the body.

The amount of distance along the arc of the circular path traveled from point A to point B is

$$d = r\theta \tag{14.1}$$

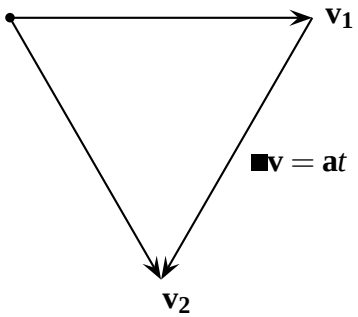


Figure 14.2: Resultant change in velocity from circular motion.

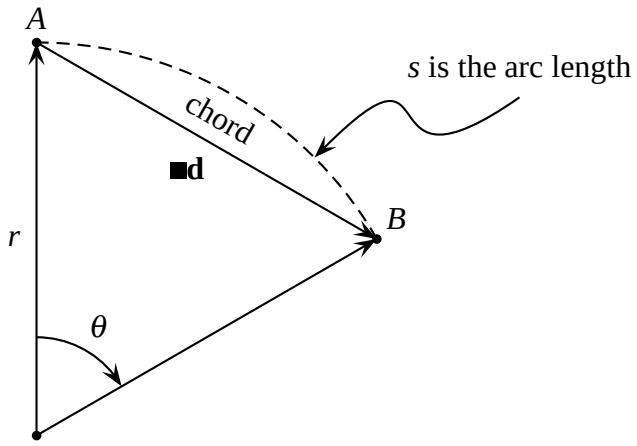


Figure 14.3: Circular arc traveled by moving from point *A* to point *B*.

where r is the radius and θ is the angular displacement in radians. If we desire the average velocity over the arc, then we can divide both sides of Eq. (14.1) by Δt to get the change in distance with respect to the change in time. This defines average velocity:

$$\begin{aligned} \frac{\Delta d}{\Delta t} &= \frac{r\Delta\theta}{\Delta t} \\ v &= r\omega \end{aligned} \tag{14.2}$$

where ω is the angular velocity in radians per second. If velocity is changing with respect to time, we can write a formula for the average acceleration:

$$\begin{aligned} \frac{\Delta v}{\Delta t} &= \frac{r\Delta\omega}{\Delta t} \\ a &= r\alpha \end{aligned} \tag{14.3}$$

where α is the angular acceleration in radians per second per second.

14.2 Lateral Acceleration

In Figure 14.3, r is the radius of the path followed by the body, and ■**d** is the displacement of the body after time t. This time is the same time as in Figure 14.2. If we assume the time interval to be quite short, then the length of the arc of the circle from *A* to *B* will be almost the same as the length of the chord from *A* to *B*. This assumption is a simple form of "taking the limit," a simple operation commonly used in math. We can then express Δd in the form of a constant velocity equation:

$\Delta d = v\Delta t$. If we look now at Figures 14.2 and 14.3, we can see that the triangles are similar. Let's look at the relationship that exists between these two triangles: Both triangles are isosceles triangles, since $\mathbf{v_1} = \mathbf{v_2}$ and $r = r$. Taking into account the relationship that exists between similar triangles, we may write the ratio appearing below. We will drop the subscripts on the velocities and deal with the magnitude (speed) only, as the speed of each velocity is the same. This gives us:

$$\frac{r}{vt} = \frac{v}{at} \tag{14.4}$$

where r is the radius of the path followed by the body, vt is the distance traveled by the body, v is the speed of the body, and at is the change in speed of the body. In order to simplify this ratio, we begin by cross-multiplying the terms. This gives us:

$$r(at) = v(vt) = v^2t \tag{14.5}$$

Now we may divide both sides of the equation by t, giving us:

$$ar = v^2 \tag{14.6}$$

Let's now divide both sides of the equation by r, giving us a as the sole term on the left side:

$$a = \frac{v^2}{r} \tag{14.7}$$

This is the form of the equation that we are looking for, since it relates the acceleration of the body to the radius of the path and the squared speed of the body. The acceleration just described in Eq. (14.7) is called *centripetal* acceleration and is directed in toward the center of the circular path followed by the body. There is a force associated with this acceleration, called centripetal force, that holds the body in its path.

In addition to centripetal acceleration and centripetal force, there is another *perceived* acceleration and another associated pseudo force, called *centrifugal* force, acting on the body. In reality, there is no such thing as centrifugal force. This perceived centrifugal force has the same magnitude as centripetal force, but has the opposite direction. The perception of centrifugal force is a consequence of Newton's First Law, which states that things in motion tend to stay in motion. So when you are riding in a car that is going around a curve and you feel like you are being forced toward the side of the car, you are actually just trying to maintain your current velocity and the car is turning into you. The real force in the scenario is centripetal force since you are accelerating toward the center

of the turn.

How can we say that a vehicle is traveling in a circular path when it goes off the road? If a vehicle is leaving critical speed yaw marks, we know that its brakes are not being locked, since the tires are rotating as well as sliding. The force trying to retard the vehicle's forward motion is small compared to the force turning the vehicle. Typical deceleration values for a longitudinal deceleration in a critical speed yaw range from 0.00g to at most -0.35g, with normal around a -0.20g on dry pavement. Consequently, in the first hundred or so feet of a measurable scuff mark, the speed of the vehicle is reduced little. We must remember that the equation we just developed deals with a constant speed. Since the vehicle can no longer be steered and since there are no other forces acting to change the direction of the vehicle, it must then follow a constant radius path.

Example 14.1. A vehicle is turning with a constant radius of 400 ft (122 m). Its speed is 45 mph (73 kph).

a.) What is the lateral acceleration of the vehicle?

b.) What is the lateral acceleration factor of the vehicle?

c.) What is the force the driver experiences during the maneuver if he weighs 180 lb (81.63 kg)?

Solution

a.) The lateral acceleration of the vehicle is determined by applying Eq. (14.7). To make sense of the dimensions of acceleration, we must first convert the speeds into the same units as the radius. Velocity must be expressed in fps or m/sec.

US	SI
$v = 1.466S$	$v = 0.277S$
$= 65.97\,\text{fps}$	$= 20.22\,\text{mps}$

Substitute this value into Eq. (14.7):

US	SI
$a = \frac{v^2}{r}$	$a = \frac{v^2}{r}$
$= \frac{65.97^2}{400}$	$= \frac{22.22^2}{122}$
$= 10.88\,\text{ft/sec}^2$	$= 3.35\,\text{m/sec}^2$

b.) The lateral acceleration factor is simply the actual lateral acceleration, determined in Step 1 above, divided by the acceleration due to gravity. Both systems of measurement should give the same acceleration factor.

$f = \frac{a}{g}$	$f = \frac{a}{g}$
$= \frac{10.88}{32.2}$	$= \frac{3.35}{9.81}$
$= 0.34$	$= 0.34$

c.) The lateral force acting on the driver comes from Newton's Second Law, $F = ma$:

$F = \frac{w}{g}a$	$F = ma$
$= fw$	$= (81.63)3.35$
$= 0.34(180)$	$= 274.46\,\text{N}$
$= 61.2\,\text{lb}$	$= 27.90\,\text{kgf}$

Example 14.2. A vehicle is traveling in a radius of 100 m (327 ft). Its lateral acceleration factor is 0.75. What is its speed in kph (mph) and what is its speed in m/s (ft/s)?

Solution

We solve this problem by rewriting Eq. (14.7) in terms of v:

$$a = \frac{v^2}{r}$$
$$ar = v^2$$
$$v = \sqrt{ar}$$

Recall that $a = fg$, so:

$$v = \sqrt{fgr} \tag{14.8}$$

US	SI
$v = \sqrt{(0.75)(32.2)(327)}$	$v = \sqrt{(0.75)(9.81)(100)}$
$v = 88.86\,\text{ft/s}$	$v = 27.11\,\text{m/s}$
$S = \frac{88.86}{1.466}$	$S = 3.6(27.11)$
$S = 60.61\,\text{mph}$	$S = 97.60\,\text{kph}$

14.3 Rotational Motion

Just as objects can undergo translational motion, they can also undergo rotary motion. A motorcycle stays upright because of the gyroscopic effect of the wheels. Sometimes, vehicles will spin to rest after striking a pole. If we develop a basic understanding of rotational mechanics, we can examine problems involving such rotating bodies.

There are rotational terms analogous to the familiar translational terms of displacement, velocity, acceleration, mass, force, momentum, and energy. The equations for the rotational quantities have very similar form to those of the translational equations. We use mostly Greek letters when dealing

with rotational quantities.

14.3.1 Angular Displacement

Angular displacement is denoted by θ and is measured in radians (refer to Section 5.2.2.4 on page 5-11 to review radian measure). The angular displacement is analogous to translational displacement d.

14.3.2 Angular Velocity

We use the symbol ω (Greek letter omega) to indicate angular velocity in radians per second. Angular velocity is defined as the time rate of change in angular displacement. The average angular velocity is:

$$\omega = \frac{\Delta\theta}{\Delta t} \tag{14.9}$$

Angular velocity is analogous to translational velocity.

14.3.3 Angular Acceleration

Angular acceleration is typically called α (Greek letter alpha) and is measured in radians per second per second. It is defined as the change in angular velocity with respect to a change in time. The average angular acceleration is:

$$\alpha = \frac{\Delta\omega}{\Delta t} \tag{14.10}$$

If we desire instantaneous quantities of angular motion, the derivative must be used, so:

$$\omega = \frac{d\theta}{dt} \tag{14.11}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \tag{14.12}$$

14.4 Mass Moment of Inertia

The mass moment of inertia of a body is a measure of the resistance that the body has to angular acceleration. As such, the moment of inertia is the rotational analog to mass since mass is the resistance of a body to translational acceleration. The moment of inertia is both a function of the mass and shape of the body. The mass moment of inertia is computed using the integral:

$$I = \int_m r^2 \, dm \tag{14.13}$$

where m is the whole mass and r is the distance of the differential mass dm from the axis of rotation. As such, the quantity I also depends on the axis of rotation. In other words, the mass moment of inertia depends on the location of the point of rotation.

Example 14.3. Determine the mass moment of inertia of a homogeneous rectangular plate shown in Figure 14.4.

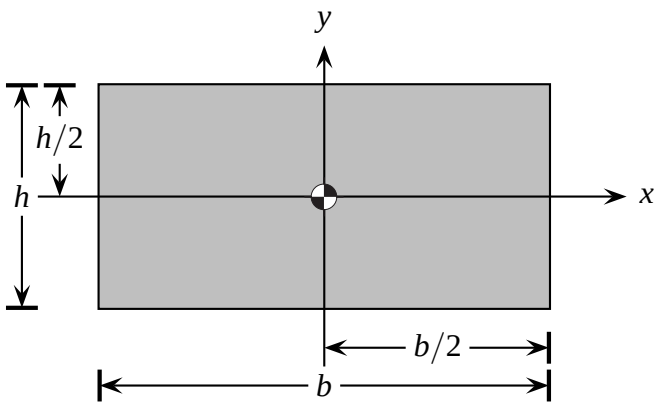


Figure 14.4: A rectangular plate with a constant density, ρ, and constant thickness, t.

There will be three axes of rotation to consider when solving this problem. Figure 14.4 shows the plate with the x-axis and y-axis. The z-axis follows the right-hand rule and is coming out of the page.

Let's start with rotation about the x-axis. The distance from the axis is simply the y coordinate of the plate. Using this, we can substitute y^2 for r^2 in Eq. (14.13):

$$I_x = \int y^2 \, dm$$

If the plate is homogeneous, then the density and thickness is constant. Also, since the plate is a rectangle, the width is constant. This means we can write the differential mass as the density times

the differential volume:

$$dm = \rho t b\, dy$$

where ρ is the density, t is the thickness, b is the width, and dy is the differential height. Therefore, the integral becomes:

$$I_x = \rho t b \int_{-h/2}^{h/2} y^2\, dy$$

The limits of integration are the boundaries of the plates. The solution proceeds as follows:

$$\begin{aligned} I_x &= \rho t b \left(\frac{y^3}{3} \Big|_{-h/2}^{h/2} \right. \\ &= \rho t b \left(\frac{(h/2)^3}{3} - \frac{(-h/2)^3}{3} \right) \\ &= \rho t b \left(\frac{h^3}{24} + \frac{h^3}{24} \right) \\ &= \rho t b \left(\frac{h^3}{12} \right) \end{aligned}$$

We can factor out an h and recognize that the product $\rho t b h$ is the total mass of the plate. Hence,

$$I_x = \frac{1}{12} m h^2 \tag{14.14}$$

A similar procedure results in:

$$I_y = \frac{1}{12} m b^2 \tag{14.15}$$

Determining the mass moment of inertia about the z-axis requires the use of a double integral and the fact that $r^2 = x^2 + y^2$. The result (without derivation) is:

$$I_z = \frac{1}{12} m (b^2 + h^2) \tag{14.16}$$

If the object is made up of several simple parts, then the integral can be expressed as a summation:

$$I = \sum_{i=1}^{n} m_i r_i^2 \tag{14.17}$$

where n is the total number of simple parts. A more detailed discussion, as well as tables, on the mass moment of inertia can be found in most introductory dynamics texts used in an engineering program. Due to the complexity of the real world, the actual determination of mass moment of inertia is best done experimentally.

For vehicles, performing this integral is nearly impossible and measuring these quantities requires a significant endeavor. There are three axes of rotation of a rigid body (like a car) which we call yaw, pitch and roll. (See Figure 8.1 on page 8-2). The easiest way to determine these properties is to look them up in a database based on experimental data.

If the vehicle is not in your database, Garrott *et. al. [8]* published some regression equations to approximate the inertial parameters.

Regression equations for cars:

$$I_{yaw} = 1.03w - 1206 \tag{14.18}$$

$$I_{pitch} = 0.99w - 1149 \tag{14.19}$$

$$I_{roll} = 0.18w - 150 \tag{14.20}$$

Regression equations for light trucks:

$$I_{yaw} = 1.03w - 1343 \tag{14.21}$$

$$I_{pitch} = 1.12w - 1657 \tag{14.22}$$

$$I_{roll} = 0.22w - 235 \tag{14.23}$$

These regression equations require the weight w to be in pounds and the resulting units are lb-ft-second^2. Also, these equations are not valid for mid-engine cars.

Example 14.4. Given a mass moment of inertia in lb-ft-second^2, what is its value in kg-m^2?

Solution

This is a problem involving conversion of units. Let's say we have $I = 1$ lb-ft-second^2. Look up conversion factors from Table 5.2 on page 5-6 to change units.

$$1\,\text{lb-ft-sec}^2 \left(\frac{1\,\text{slug}}{1\frac{\text{lb-sec}^2}{\text{ft}}}\right)\left(\frac{14.59\,\text{kg}}{1\,\text{slug}}\right)\left(\frac{1\,\text{m}}{3.28\,\text{ft}}\right)\left(\frac{1\,\text{m}}{3.28\,\text{ft}}\right) = 1.356\,\text{kg-m}^2$$

What if we are given the moment of inertia in slug-ft^2?

$$1\,\text{slug-ft}^2 \left(\frac{14.59\,\text{kg}}{1\,\text{slug}}\right)\left(\frac{1\,\text{m}}{3.28\,\text{ft}}\right)\left(\frac{1\,\text{m}}{3.28\,\text{ft}}\right) = 1.356\,\text{kg-m}^2$$

As we can see,

$$1\,\text{lb-ft-sec}^2 = 1\,\text{slug-ft}^2$$

14.4.1 The Parallel Axis Theorem

If we know the moment of inertia of a body about one axis and wish to determine the moment of inertia about another parallel axis, we use the following formula:

$$I = I_G + ml^2 \tag{14.24}$$

where I_G is the known moment of inertia,
m is the mass of the object, and
l is the distance between the two axes.

This makes sense if you think of spinning a ball. It easily spins on its axis. If you take the ball and tie it to a string and make the ball spin around your finger, it takes more effort because the moment of inertia has increased. The axis of rotation for two-dimensional objects is the z-axis, which is perpendicular to the planar surface of the page. Therefore, the distance between the two axes of rotation is the length l between the two points in 2-D.

Example 14.5. Determine the yaw moment of inertia about the king pin of the empty semi trailer that weighs 13,425 lb (6102 kg), as shown in Figure 14.5.

The following information is from reference [9]:

$$I_G = 110,739\,\text{lb-ft-sec}^2$$

for an empty 48-ft box van trailer that weighs 13,800 lbs. The subscript $_G$ indicates that the moment of inertia assumes rotation about the center of gravity. Since the source is in US units, a conversion to SI must be done using the conversion factor shown in Example 14.4:

$$I_G = 110,739\,\text{lb-ft-sec}^2 \left(1.356 \frac{\text{kg-m}^2}{\text{lb-ft-sec}^2}\right)$$
$$I_G = 150,162\,\text{kg-m}^2$$

If the location of rotation is different from the center of mass, then, by the Parallel Axis Theorem, the yaw moment of inertia changes to

$$I = I_G + mh^2$$

where I is the yaw moment of inertia about the point of rotation,
I_G is the mass moment of inertia about the center of mass,
m is the mass of the object, and
h is the distance from the center of mass to the point of rotation.

The center of mass of the trailer is slightly off-center, but not enough to make a difference when computing the value of h using the Pythagorean Theorem. The value of h, as seen in Figure 14.5, is 27.3 feet (8.32 m). So, by the Parallel Axis Theorem,

US	SI
$I = I_G + \left(\frac{w_T}{g}\right) h^2$	$I = I_G + mh^2$
$= 110,739 + \left(\frac{13,425}{32.2}\right) 27.3^2$	$= 150,162 + (6102) 8.32^2$
$= 421,470\,\text{lb-ft-sec}^2$	$= 572,557\,\text{kg-m}^2$

14.4.2 Radius of Gyration

The radius of gyration is a useful quantity because it describes the radius at which all of the mass could be concentrated to have an equivalent moment of inertia. If we substitute the radius of gyration k for r in Eq. (14.13) and recognize that k is fixed:

$$I = k^2 \int_m dm \tag{14.25}$$

$$I = mk^2 \tag{14.26}$$

Example 14.6. Determine the square of the radius of gyration of the trailer in Example 14.5.

Use Eq. (14.26) and solve for k^2:

$$k^2 = \frac{I}{m} \tag{14.27}$$

US	SI
$k^2 = \dfrac{I}{\left(\frac{w}{g}\right)}$	$k^2 = \dfrac{I}{m}$
$= \dfrac{421,470\,\text{lb-ft-sec}^2\left(32.2\,\frac{\text{ft}}{\text{sec}^2}\right)}{13,425\,\text{lb}}$	$= \dfrac{572,557\,\text{kg-m}^2}{6102\,\text{kg}}$
$= 1010.90\,\text{ft}^2$	$= 93.83\,\text{m}^2$

14.5 Newton's Second Law for Rotation

The concept of torque was first introduced when we discussed cross product in Section 6.6.3 and explained further in Section 10.7. The definition of torque uses the cross product:

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} \tag{14.28}$$

where $\boldsymbol{\tau}$ is the torque vector, **r** is the vector pointing from the point of rotation to the applied force, and **F** is the applied force vector. Basically, torque is produced when a force is applied against a lever, such as pushing against a wrench to remove a nut. When calculating torque, the force must be acting against the lever at a right angle. What if the force is not acting against the lever at a right angle? This is the case in Figure 14.6. Force F is acting against lever r at some angle θ. What we have to do is calculate the perpendicular length of the lever arm against which the force is acting.

As seen in Figure 14.6, the effective lever arm, L, is the side opposite angle θ. Therefore, we can define its length by:

$$L = r\sin\theta \tag{14.29}$$

If we substitute this expression for L in the basic torque equation, we get:

$$\tau = Fr\sin\theta \tag{14.30}$$

Torque is the effectiveness of a force to produce a rotation about an axis. It is analogous to force and is measured in ft-lb or inch-lb in the US system and N-m in the SI system. A moment (M) is the same thing as a torque (τ). Because torque and force are analogous, as are mass and moment of inertia, as well as translational acceleration and angular acceleration, it follows that

$$\boxed{\tau = I\alpha} \tag{14.31}$$

is Newton's Second Law for rotation.

14.6 Changing Torque and Gear Ratios

Machines use gears, belts, and chains to change torque output. These simple machine components are required because motors typically do not produce enough torque from their shafts. Also, sufficient torque is usually produced at high speeds and this needs to be controllable. The solution to this problem is a transmission.

We discussed torque in Section 10.7 and a review would be prudent. In our remaining discussion, we'll assume the torque vector is perpendicular to the plane that comprise the radius and force

vector. This means we can write all the vectors as magnitudes as follows:

$$\tau = Fr \tag{14.32}$$

where τ is the magnitude of the torque vector,
F is the magnitude of the force vector, and
r is the length of the radius vector.

Torque is multiplied (either increased or decreased) by the ratio of the size of the machine components. For chain sprockets and gears, the ratio of the teeth is used. For pulleys, the ratio of diameters is used.

Example 14.7. A chain connects a 25-tooth driver sprocket to a 65-tooth driven sprocket. What is the change in the driving torque?

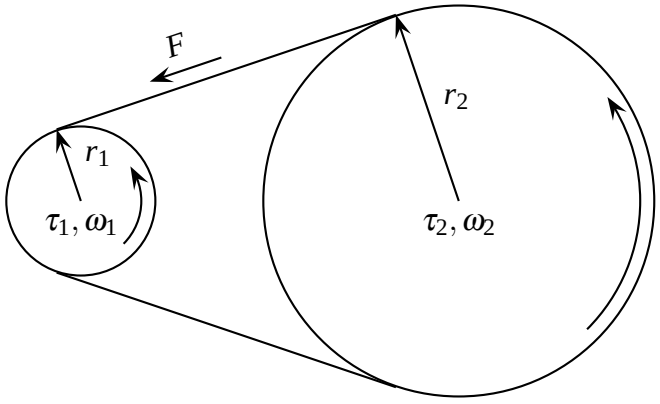


Figure 14.7: A diagram of a chain connecting two sprockets. The tension in the chain is felt on both sprockets. The smaller sprocket is driving the larger sprocket.

Lets look at Figure 14.7: We can see that the force acting on sprocket #2 comes from the tension in the chain. This tension is generated by sprocket #1 according to

$$F = \frac{\tau_1}{r_1}$$

Since this force is common to both sprockets,

$$F = \frac{\tau_1}{r_1} = \frac{\tau_2}{r_2}$$

By cross multiplying we obtain our ratio:

$$\frac{\tau_1}{\tau_2} = \frac{r_1}{r_2} \tag{14.33}$$

This is a nice result, but we do not know the actual radii of the sprockets. We do, however, know the number of teeth. An important fact about gears and sprockets is that they have a circumference that is some multiple of the pitch. The pitch is the spacing between the gear teeth or the sprocket teeth. The pitch has to be the same on both components in order to mesh. We also know that the circumference is $C = 2\pi r$. This leads to:

$$C_1 = n_1 p = 2\pi r_1$$

which says that the number of teeth times the pitch is equal to the circumference. We can isolate p and set #1 equal to #2 due to the common pitch:

$$\frac{2\pi r_1}{n_1} = p = \frac{2\pi r_2}{n_2}$$

We can cross multiply and cancel common terms to arrive at the relation

$$\frac{r_1}{r_2} = \frac{n_1}{n_2} \tag{14.34}$$

Eq. (14.34) can be substituted into Eq. (14.33) to find the final torque multiplier:

$$\begin{aligned}\frac{\tau_1}{\tau_2} &= \frac{n_1}{n_2} \\ &= \frac{25}{65} \\ &= 0.384\end{aligned}$$

So, if 1 ft-lb of torque was supplied to shaft #1, shaft #2 would feel 2.6 ft-lb of torque.

Example 14.8. Let's say the rotation rate of shaft #1 in Figure 14.7 is 25 rpm while producing a torque of 12 N-m. What is the power produced at shaft #2?

We are still using the same gear ratio of 1:2.6. The rotation rate of the first sprocket needs to be translated to the rotation rate of the second sprocket. We also have enough information to determine

the power of the first shaft:

$$\begin{aligned} P_1 &= \tau_1 \omega_1 \\ P_1 &= (12\,\text{N-m}) \left(\frac{2500\,\text{rev}}{1\,\text{min}} \right) \left(\frac{1\,\text{min}}{60\,\text{sec}} \right) \left(\frac{2\pi\,\text{rad}}{1\,\text{rev}} \right) \\ P_1 &= 3141\,\text{Watts} \end{aligned}$$

The speed of the chain when it meets the sprocket must be

$$v = r\omega$$

and this velocity is common to both sprockets, so

$$r_1 \omega_1 = v = r_2 \omega_2$$

This gives us the inverse relation:

$$\frac{r_1}{r_2} = \frac{\omega_2}{\omega_1} \tag{14.35}$$

Between Eqs. (14.33-14.35) we can see that

$$\begin{aligned} P_2 &= \tau_2 \omega_2 \\ P_2 &= \left(\tau_1 \frac{n_2}{n_1} \right) \left(\omega_1 \frac{n_1}{n_2} \right) \\ P_2 &= \tau_1 \omega_1 \\ P_2 &= P_1 \end{aligned}$$

We conclude that if the power remains the same, then as the torque increases, the speed must decrease. In reality, there are energy losses in the coupling and some power is lost in the transmission. This means that the value reported for shaft #2 is a theoretical maximum.

Sometimes the amount of torque increase is too much to be accommodated by one set of shafts. In this case, multiple components can be linked together to achieve the goals of changing the torque and saving space. To calculate the gear ratio in this case, simply multiply all ratios in the system together.

Example 14.9. A transmission has a 10:1 gear ratio and the differential ratio is 4.10:1. What is the overall gear ratio?

The overall gear ratio is the product of the gear ratio of the transmission and the differential, which is 41:1. Therefore, the axle torque is 41 times greater than the engine torque and the engine speed is 41 times the axle rotation rate. These relations can be used to determine possible speeds based on gear ratios.

14.7 Rotational Kinetic Energy

As in translational motion, work done by rotation is the scalar product of torque and angular displacement.

$$W = \tau\theta \tag{14.36}$$

The same derivation can be done as shown in Section 13.5 on page 13-15 to arrive at the result for kinetic energy due to rotation:

$$KE = \frac{1}{2}I\omega^2 \tag{14.37}$$

The total kinetic energy is the sum of the translational kinetic energy and the rotational kinetic energy.

A similar discussion for power leads to the result that

$$P = \tau\omega \tag{14.38}$$

which says that power is the scalar product of torque and angular velocity.

Example 14.10. A vehicle has a yaw moment of inertia of 2000 lb-ft-sec^2 (2710 kg-m^2). It is spinning at 7 radians per second (400°). How much rotational kinetic energy does the vehicle possess?

Solution

Use Eq. (14.37) to determine the kinetic energy of the spinning vehicle:

US	SI
$KE = \frac{1}{2} I\omega^2$	$KE = \frac{1}{2} I\omega^2$
$= \frac{1}{2}(2000)(7)^2$	$= \frac{1}{2}(2710)(7)^2$
$= 49,000\,\text{ft-lb}$	$= 66,395\,\text{J}$

If this vehicle weighs 3000 lb (1360 kg) and is going 60 mph (97 kph), how much translational kinetic energy does it have?

US	SI
$KE = \frac{wS^2}{30}$	$KE = \frac{mS^2}{26}$
$= \frac{3000(60)^2}{30}$	$= \frac{1360(97)^2}{26}$
$= 360,000\,\text{ft-lb}$	$= 492,163\,\text{J}$

What is the total energy of this spinning and translating vehicle?

$KE_{total} = KE_{trans} + KE_{rot}$	$KE_{total} = KE_{trans} + KE_{rot}$
$= 360,000 + 49,000$	$= 492,163 + 66,395$
$= 409,000\,\text{ft-lb}$	$= 558,558\,\text{J}$

If all of the vehicle's kinetic energy was translational, how fast would it be going?

$$S = \sqrt{\frac{30KE}{w}}$$
$$= \sqrt{\frac{30(409,000)}{3000}}$$
$$= 63.95\,\text{mph}$$

$$S = \sqrt{\frac{26KE}{m}}$$
$$= \sqrt{\frac{26(558,558)}{1360}}$$
$$= 103.33\,\text{kph}$$

14.8 Angular Momentum

Angular momentum, denoted with Q, is defined as the product of the mass moment of inertia and the angular velocity.

$$Q = I\omega \tag{14.39}$$

It can also be shown (without derivation) that

$$impulse = \tau\Delta t \tag{14.40}$$

and change in angular momentum produced by an external torque is equal to the angular impulse:

$$\tau\Delta t = I(\omega_f - \omega_o) \tag{14.41}$$

If $\omega_o = 0$ then

$$\tau\Delta t = I\omega \tag{14.42}$$

Rectilinear Motion	Rotational Motion
$d = vt$	$\theta = \omega t$
$d = v_o t + \frac{1}{2} a t^2$	$\theta = \omega_o t + \frac{1}{2} \alpha t^2$
d	θ
v	ω
a	α
$t = \frac{\Delta v}{a}$	$t = \frac{\Delta \omega}{\alpha}$
$v_{avg} = \frac{v_o + v_f}{2}$	$\omega_{avg} = \frac{\omega_o + \omega_f}{2}$
$v = \sqrt{v_o^2 + 2ad}$	$\omega = \sqrt{\omega_o^2 + 2\alpha\theta}$
$d = \frac{v^2}{2a}$	$\theta = \frac{\omega^2}{2\alpha}$

Table 14.6: Time–distance equivalents between rotational and translational motion.

Rectilinear Motion		Rotational Motion		Vector	Scalar
Force	$\mathbf{F}$	Torque	$\boldsymbol{\tau}$	✓	
Mass	m	Moment of Inertia	I		✓
Displacement	$\mathbf{d}$	Angular Displacement	θ	✓	
Velocity	$\mathbf{v} = \frac{\blacksquare\mathbf{d}}{\Delta t}$	Angular Velocity	$\omega = \frac{\Delta\theta}{\Delta t}$	✓	
Acceleration	$\mathbf{a} = \frac{\blacksquare\mathbf{v}}{\Delta t}$	Angular Acceleration	$\alpha = \frac{\Delta\omega}{\Delta t}$	✓	
Work	$W = \mathbf{F} \cdot \mathbf{d}$	Work	$W = \tau\theta$		✓
Kinetic Energy	$KE = \frac{1}{2}mv^2$	Kinetic Energy	$KE = \frac{1}{2}I\omega^2$		✓
Power	$P = \mathbf{F} \cdot \mathbf{v}$	Power	$P = \tau\omega$		✓
Linear Momentum	$\mathbf{P} = m\mathbf{v}$	Angular Momentum	$Q = I\omega$	✓	
Impulse	$impulse = F\Delta t$	Impulse	$impulse = \tau\Delta t$	✓	

Table 14.7: A comparison of equations describing translational motion with rotational motion. Each quantity is either a vector or a scalar.

Example 14.11. A vehicle that has a moment of inertia of 2000 lb-ft-sec^2(2710 kg-m^2) impacts a pole. The lever arm L for the impact is 1 foot (0.304 m) The collision force is 30,000 lb (133,500 N) and this force acts for 0.10 seconds. What is the angular velocity of the car after impact?

Solution

This problem is solved using the concept of angular impulse and momentum, as shown in Eq. (14.42). Since the initial impact was assumed to be without spin, $\omega_o = 0$. Also, we can write the torque as a force times the lever arm $\tau = FL$.

$$\tau \Delta t = I\omega \tag{14.43}$$

$$\omega = \frac{\tau \Delta t}{I} \tag{14.44}$$

$$\omega = \frac{FL\Delta t}{I} \tag{14.45}$$

US	SI
$\omega = \dfrac{30,000(1)(0.10)}{2000}$	$\omega = \dfrac{133,500(0.304)(0.10)}{2710}$
$= 1.5\,\text{rad/sec}$	$= 1.5\,\text{rad/sec}$

Notice that both systems of measurement have the same units for angular measure. How much rotational kinetic energy does this car have after the impact?

$KE = \dfrac{1}{2}I\omega^2$	$KE = \dfrac{1}{2}I\omega^2$
$= \dfrac{1}{2}(2000)(1.5)^2$	$= \dfrac{1}{2}(2710)(1.5)^2$
$= 2250\,\text{ft-lb}$	$= 3048\,\text{J}$

If this 3000-lb (1360-kg) vehicle had a true post impact speed of 35 mph (56 kph), how much faster would it be going if it was not spinning?

US

$$\begin{aligned} KE_{total} &= KE_{trans} + KE_{rot} \\ &= \frac{wS^2}{30} + 2250 \\ &= \frac{3000(35)^2}{30} + 2250 \\ &= 124,750\,\text{ft}\cdot\text{lb} \end{aligned}$$

$$\begin{aligned} S &= \sqrt{\frac{30KE}{w}} \\ &= \sqrt{\frac{30(124,750)}{3000}} \\ &= 35.31\,\text{mph} \end{aligned}$$

Answer: 0.31 mph faster

SI

$$\begin{aligned} KE_{total} &= KE_{trans} + KE_{rot} \\ &= \frac{mS^2}{26} + 3048 \\ &= \frac{1363(56)^2}{26} + 2250 \\ &= 166,286\,\text{J} \end{aligned}$$

$$\begin{aligned} S &= \sqrt{\frac{26KE}{m}} \\ &= \sqrt{\frac{26(166,286)}{1363}} \\ &= 56.38\,\text{kph} \end{aligned}$$

Answer: 0.38 kph faster

The difference in speed from accounting for the rotational kinetic energy is insignificant in this example. Many crashes are this way unless rotational rates are very high and/or the moment of inertia is very large. Determining what is large requires an analysis both with and without rotation to see if there is a difference in the final speed calculation.

Example 14.12. A vehicle has a moment of inertia of 2000 lb-ft-sec^2 (2710 kg-m^2). It has a collision force of 40,000 lb (178,000 N) acting on it for 0.10 seconds, as shown in Figure 14.8.

a.) What is the torque placed on the vehicle?

b.) What is the angular velocity of the vehicle after the impact?

Solution

We must first determine the length of the lever arm h by analyzing the similar triangles shown in Figure 14.9. In this case, we need the tangent function to determine the length x, and then we use the sine function to solve for h.

US	SI
$\tan 20^\circ = \dfrac{1}{x}$	$\tan 20^\circ = \dfrac{0.304}{x}$
$x = \dfrac{1}{\tan 20^\circ}$ $= \dfrac{1}{0.363}$ $= 2.75\,\text{ft}$	$x = \dfrac{0.304}{\tan 20^\circ}$ $= \dfrac{0.304}{0.363}$ $= 0.837\,\text{m}$
$L = 8 - x$ $= 5.25\,\text{ft}$	$L = 2.43 - x$ $= 1.59\,\text{m}$

Calculate h:

$\dfrac{h}{L} = \sin\theta$ $h = L\sin\theta$ $= 5.25(0.342)$ $= 1.79\,\text{ft}$	$\dfrac{h}{L} = \sin\theta$ $h = L\sin\theta$ $= 1.59(0.342)$ $= 0.543\,\text{m}$

US	SI

Calculate torque:

$$\tau = Fh$$

US	SI
$\tau = 40,000(1.79)$ $= 71,600\,\text{lb-ft}$	$\tau = 178,000(0.543)$ $= 96,654\,\text{N-m}$

Calculate angular velocity:

$$\omega = \frac{\tau \Delta t}{I}$$

US	SI
$\omega = \frac{71,600(0.10)}{2000}$ $= 3.5\,\text{rad/sec}$	$\omega = \frac{96,654(0.10)}{2710}$ $= 3.5\,\text{rad/sec}$

Example 14.13. A 3000-lb (1360-kg) vehicle is traveling at 45 mph (72.4 kph) up a long, smooth ramp that is inclined at 20°.* The wheelbase of the car is 108 in (2.74 m). The center of mass of the car is 45 in (1.14 m) behind the front axle and 18 in (0.457 m) off the ground. The rolling radius of the rear tires, r_R, is 13 inches (0.33 m). The pitch moment of inertia is 2000 lb-ft-sec^2 (2710 kg-m^2). What is the pitch angular velocity of the car when the rear wheels just clear the ramp? What is the pitch displacement of the car when it clears the ramp?

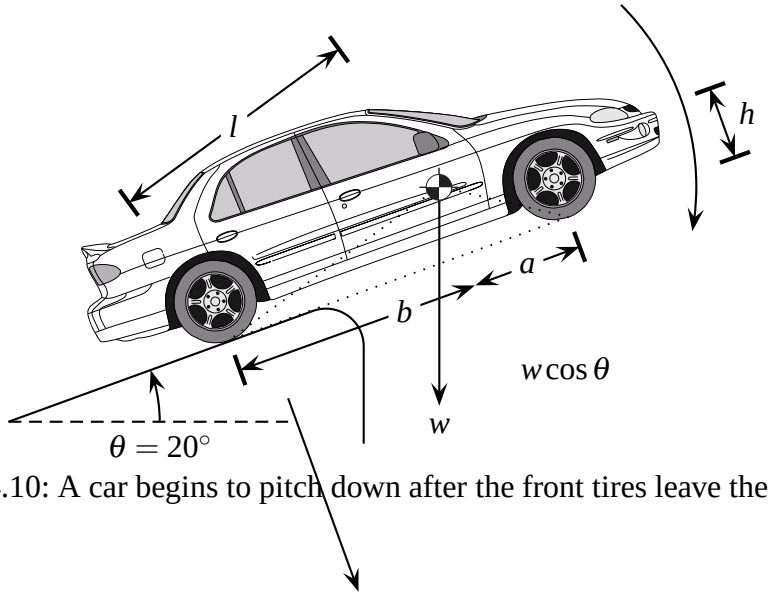


Figure 14.10: A car begins to pitch down after the front tires leave the ramp.

A sketch shown in Figure 14.10 will help us understand the problem. The solution steps are as follows:

a.) Determine the time to clear the ramp.

US	SI
$t = \frac{d}{v} = \frac{d}{1.466S}$	$t = \frac{d}{v} = 3.6\left(\frac{d}{S}\right)$
$= \frac{9}{1.466(45)}$	$= 3.6\left(\frac{2.74}{72.4}\right)$
$= 0.136\,\text{sec}$	$= 0.136\,\text{sec}$

b.) Calculate the torque about the rear axle when the lever arm is equal to b and the perpendicular force producing the torque is $w\cos\theta$:

*The ramp must be long and smooth so that the initial pitch velocities are zero. If the ramp is too short or changes profile, additional forces may induce torque.

$$\tau = b(w\cos\theta)$$
$$= 5.25(3000)(0.939)$$
$$= 14,789\,\text{lb-ft}$$

$$\tau = b(mg\cos\theta)$$
$$= 1.6(1360)(9.81)(0.939)$$
$$= 20,023\,\text{N-m}$$

c.) Use the Parallel Axis Theorem to calculate the moment of inertia around the rear axle. The length from the rear axle to the center of mass is determined using the Pythagorean Theorem, $l^2 = b^2 + (h - r_R)^2$.

US

$$r_R = 1.08 \text{ ft}$$

$$I = I_G + ml^2$$
$$= I_G + \frac{w}{g}(b^2 + (h - r_R)^2)$$
$$= 2000 + \frac{3000}{32.2}(5.25^2 + (1.5 - 1.08)^2)$$
$$= 4584\,\text{slug-ft}^2$$

SI

$$r_R = 0.33\,\text{m}$$

$$I = I_G + ml^2$$
$$= I_G + m(b^2 + (h - r_R)^2)$$
$$= 2710 + 1360(1.6^2 + (0.457 - 0.33)^2)$$
$$= 6213\,\text{kg-m}^2$$

d.) Calculate the angular velocity using the principle of angular impulse and momentum from Eq. (14.42), since the initial angular velocity is zero.

$$\omega = \frac{\tau \Delta t}{I}$$
$$= \frac{14,789(0.136)}{4584}$$
$$= 0.43\,\text{rad/sec}$$

$$\omega = \frac{\tau \Delta t}{I}$$
$$= \frac{20,023(0.136)}{6213}$$
$$= 0.43\,\text{rad/sec}$$

e.) Determine the pitch angle when the rear tires clear the ramp by first calculating an average angular acceleration from Newton's Second Law for rotation:

US	SI
$\tau = I\alpha$	$\tau = I\alpha$
$\alpha = \frac{\tau}{I}$	$\alpha = \frac{\tau}{I}$
$= \frac{14,789}{4584}$	$= \frac{20,023}{6213}$
$= 3.22\,\text{rad/sec}^2$	$= 3.22\,\text{rad/sec}^2$

The US and SI systems are the same for units of time and angular displacement, so:

$$\begin{aligned}\theta &= \frac{1}{2}\alpha t^2 \\ &= \frac{1}{2}(3.22)(0.136)^2 \\ &= 0.03\,\text{rad} \\ &= 1.7^\circ\end{aligned}$$

Let us ask an additional question: What is the minimum speed required for the vehicle to just clear the ramp without scraping, if it can pitch down no more than 10° (0.17 rad)? We can assume a constant torque for this small angular displacement. If the angular displacement was larger, we may have to adjust the torque as a function of the angle.

We begin answering this question by finding the time required to reach the maximum allowable pitch with the average angular acceleration we calculated above:

$$\begin{aligned}t &= \sqrt{\frac{2\theta}{\alpha}} \\ &= \sqrt{\frac{2(0.17)}{3.22}} \\ &= 0.32\,\text{sec}\end{aligned}$$

Now calculate speed where d is the wheelbase length in feet or meters:

US	SI
$S = \dfrac{d}{1.466t}$	$S = 3.6\dfrac{d}{t}$
$S = \dfrac{9}{1.466(0.32)}$	$S = 3.6\left(\dfrac{2.74}{0.32}\right)$
$S = 19.18\,\text{mph}$	$S = 30.82\,\text{kph}$

In this example we assumed the rotation took place about the rear axle. If the rear wheels were locked as the car went airborne, then the point of rotation would be the rear contact patch.

14.9 Eccentric Collision Analysis Using Rotational Mechanics

When a bullet vehicle strikes a semi-trailer at near right angles and rotates the trailer, we can determine the Δv of the bullet vehicle by using the concepts of rotational mechanics. Let us begin with Newton's Second Law for rotation:

$$\tau = I\alpha \tag{14.46}$$

where τ (tau) is the torque applied to a body (equivalent to force),
I is the effective mass moment of inertia of the body (equivalent to mass), and
α is the angular acceleration in radians per second per second (equivalent to acceleration).

The radian is a unit of angular measure and there are $180/\pi = 57.29$ degrees per radian.

Newton's Third Law for rotation states that every applied torque is matched by an equal but opposite torque. Using these two laws, we can derive the principle of the conservation of angular momentum.

Consider a collision between a ball and a board, as shown in Figure 14.11 on page 14-45. When the ball strikes the board, it gives the board a linear impulse. This linear impulse acts at a distance away from a fulcrum (pivot point) and becomes an angular impulse, $\tau\Delta t$, which acts to rotate an object. Since this board is free to rotate, it accelerates according to Newton's Second Law for rotation.

The ball undergoes a change in velocity, and its initial impact speed can be determined using the concept of the coefficient of restitution. Let's see how this is done mathematically.

14.9.1 Derivation of Δv from Rotational Mechanics Concepts

Begin by separating the individual bodies from the system and analyzing their free body diagram. For the ball, the relation holds that

$$F_{c1} = ma \tag{14.47}$$

which says the force imparted on the ball by the board is equal to the inertial force of the ball.

The moments (or torques) about the pivot point are equal to the angular inertia. The torque is equal to the product of the force the ball imparts on the board and the length of the lever arm used by the applied force less the net ground forces times its effective lever arm:

$$I\alpha = h_c F_{c2} - h_g F_g \tag{14.48}$$

which means:

$$F_{c2} = \frac{I\alpha + h_g F_g}{h_c}$$

By Newton's Third Law:

$$-F_{c1} = F_{c2} \tag{14.49}$$

Making the substitution of Eqs. (14.47) and (14.48) into Eq. (14.49) gives:

$$-ma = \frac{I\alpha + h_g F_g}{h_c} \tag{14.50}$$

It is convenient to define a positive Δv as:

$$\Delta v = v_i - v_f \tag{14.51}$$

However, by using this definition of Δv in the kinematic relations from Chapter 7, defining acceleration requires that Δt be defined as the initial time less the final time. This quantity will always be

negative, thus:

$$a = \frac{v_i - v_f}{t_i - t_f} \tag{14.52}$$

where the subscript $_i$ means initial and the subscript $_f$ means final. Let $t_i = 0$ and $t_f = \Delta t$ to define acceleration as

$$a = -\frac{\Delta v}{\Delta t} \tag{14.53}$$

Substitute Eq. (14.53) into Eq. (14.50) to get:

$$\begin{aligned} -m\left(-\frac{\Delta v}{\Delta t}\right) &= \frac{I\alpha + h_g F_g}{h_c} \\ m\left(\frac{\Delta v}{\Delta t}\right) &= \frac{I\alpha + h_g F_g}{h_c} \end{aligned} \tag{14.54}$$

An analogous substitution can be made for the angular acceleration:

$$\alpha = \frac{\Delta \omega}{\Delta t} = \frac{\omega_f - \omega_i}{t_f - t_i} \tag{14.55}$$

where ω_f is the post-impact angular velocity of the rotating object, and
ω_i is the pre-impact angular velocity of the rotating object.

If the initial angular velocity of the rotating object is less than the final angular velocity, then

$\Delta\omega$ is positive. Continue by substituting the value for α into Eq. (14.54):

$$m\frac{\Delta v}{\Delta t} = \frac{I\left(\frac{\Delta\omega}{\Delta t}\right) + h_g F_g}{h_c} \tag{14.56}$$

Multiply the numerator and denominator on the right-hand side of Eq. (14.56) by Δt:

$$m\frac{\Delta v}{\Delta t} = \frac{I\Delta\omega + h_g F_g \Delta t}{h_c \Delta t} \tag{14.57}$$

Multiplying both sides of the equation by $h_c\Delta t$ yields:

$$mh_c\Delta v = I\Delta\omega + h_g F_g \Delta t \tag{14.58}$$

Now solve for Δv:

$$\Delta v = \frac{I\omega + h_g F_g \Delta t}{mh_c} \tag{14.59}$$

If the ground forces are insignificant over a short Δt then:

$$\boxed{\Delta v = \frac{I\omega}{mh_c}} \tag{14.60}$$

where m is the mass of the striking object,
I is the mass (yaw) moment of inertia of the object being struck,
ω is the post impact speed of the rotating object, and
h_c is the perpendicular distance of the line of force imparted by the striking object.

This result demonstrates that we can determine the speed change of the ball if we can determine the change in angular velocity of the board after the impact. We will extend this same concept to a vehicle hitting the side of a semi-trailer or a motorcycle striking the rear of a pickup. If a semi-trailer is heavily loaded, the ground force may become significant during the collision and Eq. (14.59) may be useful.

14.9.2 The Work–Energy Theorem for Rotation

The Work–Energy Theorem states that the work done by a torque comes from kinetic energy:

$$KE = Work \tag{14.61}$$

The kinetic energy of a rotating object is:

$$KE = \frac{1}{2} I \omega^2 \tag{14.62}$$

The work done by a torque is:

$$Work = \tau \theta \tag{14.63}$$

where τ is the torque and θ (theta) is the rotational displacement in radians. Also, the definition of torque is:

$$\tau = Fh \tag{14.64}$$

where F is a force and h is the lever arm for the force. So how do we determine the force? The force, in this case, is due to friction and is equal to the weight on tires multiplied by the drag factor. The drag factor equation accounts for the amount of side slip, percentage of braking, and slope. If the direction of tire travel is perpendicular to the rolling direction of the tire, then the maximum amount of friction force is available to that tire. So,

$$F = fw \tag{14.65}$$

where f is the drag factor and w is the weight.

If we substitute Eq. (14.64) into Eq. (14.63) and invoke Eq. (14.61), then we get:

$$\frac{1}{2} I \omega^2 = Fh\theta \tag{14.66}$$

Solving for ω gives:

$$\boxed{\omega = \sqrt{\frac{2Fh\theta}{I}}} \tag{14.67}$$

where $F = fw$ from Eq. (14.65).

Example 14.14. A 1993 Honda Accord weighing 2900 lb (1318 kg) strikes the center of the rear duals of a tractor semi-trailer combination at a right angle. The evidence of the scene indicates the tractor did not move during the collision. Therefore, the kingpin can be considered rigid as the trailer rotates. The trailer rotated 4.27° around the kingpin. Use the trailer in Example 14.5. The

weight on the rear duals (axles 3 and 4) is $w_{3,4} = 9550$ lb (4340 kg).

The force used in Eq. (14.67) comes from the friction force of the rear duals (axles 3 and 4). The lever arm for this frictional force is $h = 36.4$ ft (11.09 m). The tires are sliding perpendicular to their rolling direction, so they are producing the maximum available force. A drag factor was determined to be $f = 0.60$ for the rear duals moving sideways. To determine the value of the torque, use Eq. (14.65):

US

$$\begin{aligned}\tau = Fh &= fw_{3,4}h \\ &= 0.60(9550)(36.4) \\ &= 208,572\,\text{lb-ft}\end{aligned}$$

SI

Weight must be in newtons:

$$\begin{aligned}w_{3,4} &= 9.81(4340) \\ &= 42,532\,\text{N}\end{aligned}$$

$$\begin{aligned}\tau = Fh &= fw_{3,4}h \\ &= 0.60(42,532)(11.09) \\ &= 283,008\,\text{N-m}\end{aligned}$$

The angular values of the rotation must be in radians, thus:

$$\theta = 4.27\left(\frac{\pi}{180}\right) = 0.0746\,\text{rad}$$

Substitute the values of θ and τ from above and the value of the yaw moment of inertia from Example 14.5 into Eq. (14.67):

US

$$\begin{aligned}\omega &= \sqrt{\frac{2(208,572)(0.0746)}{421,470}} \\ &= 0.271\,\text{rad/s}\end{aligned}$$

SI

$$\begin{aligned}\omega &= \sqrt{\frac{2(283,008)(0.0746)}{572,557}} \\ &= 0.271\,\text{rad/s}\end{aligned}$$

Notice that the value of ω is the same for both systems of measurement because angular measure is common to both systems.

These values of mass, yaw moment of inertia, angular velocity, and the lever arm are now used in Eq. (14.60) to compute the Δv of the Honda:

US

The mass of the Accord is computed from the definition of weight for the US system:

$$m = \frac{w}{g} = \frac{2900}{32.2} = 90.06\,\text{slugs}$$

$$\begin{aligned}\Delta v &= \frac{I\omega}{mh} \\ &= \frac{421,470(0.271)}{90.06(36.4)} \\ &= 34.86\,\text{ft/s}\end{aligned}$$

In units of miles per hour:

$$\begin{aligned}\Delta S &= \frac{\Delta v}{1.466} \\ &= 23.77\,\text{mph}\end{aligned}$$

SI

$$\begin{aligned}\Delta v &= \frac{I\omega}{mh} \\ &= \frac{572,557(0.271)}{1318(11.09)} \\ &\quad 10.61\,\text{m/s}\end{aligned}$$

In units of kilometers per hour:

$$\begin{aligned}\Delta S &= 3.6(\Delta v) \\ &= 38.19\,\text{kph}\end{aligned}$$

14.9.3 Computing Impact Speed from Δv

In order to compute the impact speed from Δv, we use the coefficient of restitution defined as:

$$\varepsilon = \frac{v_4 - v_3}{v_1 - v_2} \tag{14.68}$$

The coefficient of restitution is the ratio of the relative velocity after the collision to the relative velocity before the collision. It has a value from 0 to 1. A higher value for the coefficient of restitution leads to a lower impact speed if the value of Δv remains constant. We have a definition of Δv that is:

$$\Delta v = v_1 - v_3$$

or

$$v_3 = v_1 - \Delta v$$

which says the post-impact velocity is the pre-impact velocity less the change in velocity for an in-line collision. If the vehicle being struck was initially not moving, then $v_2 = 0$. Making this assumption, we can substitute the expression for v_3 into Eq. (14.68):

$$\varepsilon = \frac{v_4 - (v_1 - \Delta v)}{v_1 - 0}$$

Solve for v_1:

$$\begin{aligned} \varepsilon v_1 &= v_4 - v_1 + \Delta v \\ v_1 + \varepsilon v_1 &= v_4 + \Delta v \\ v_1(1+\varepsilon) &= v_4 + \Delta v \end{aligned}$$

$$v_1 = \frac{v_4 + \Delta v}{1+\varepsilon} \tag{14.69}$$

If a vehicle strikes an immovable barrier, then $v_4 = 0$ and Eq. (14.69) simplifies to:

$$v_1 = \frac{\Delta v}{1+\varepsilon} \tag{14.70}$$

Example 14.15. Determine the impact speed of the Honda in Example 14.14 if the coefficient of restitution is a.) $\varepsilon = 0.0$ and b.) $\varepsilon = 0.15$.

The value of v_4 is determined from the kinematics of rotating bodies:

$$v_4 = r\omega \tag{14.71}$$

where r is the distance between the centroid of the collision and the point of rotation. The variable of ω describes the post-impact angular velocity and is determined using Eq. (14.67). Substitute Eq. (14.71) into Eq. (14.69) to get:

$$v_1 = \frac{r\omega + \Delta v}{1+\varepsilon} \tag{14.72}$$

US	SI
a.) $\varepsilon = 0.0$	a.) $\varepsilon = 0.0$
$v_1 = \dfrac{36.4(0.271)+34.86}{1+0.0}$	$v_1 = \dfrac{11.09(0.271)+10.61}{1+0.0}$
$= 44.72\,\text{ft/s}$	$= 13.61\,\text{m/s}$
$= 30.50\,\text{mph}$	$= 49.00\,\text{kph}$
b.) $\varepsilon = 0.15$	b.) $\varepsilon = 0.15$
$v_1 = \dfrac{36.4(0.271)+34.86}{1+0.15}$	$v_1 = \dfrac{11.09(0.271)+10.61}{1+0.15}$
$= 38.88\,\text{ft/s}$	$= 11.84\,\text{m/s}$
$= 26.52\,\text{mph}$	$= 42.62\,\text{kph}$

This technique has been tested through staged crashes to provide accurate results. The recommended technique would entail using a range of at least 0.10 on the probable drag factor for the rear duals of the trailer. If we used a range of f between 0.55 and 0.65, and a range for ε between 0.00 and 0.15, then the reconstructed speeds come to between 25.45 and 31.81 mph. The data for Example 14.14 came from an actual crash test performed at IPTM for the Special Problems 2005 conference. The actual impact speed recorded for this event was 31 mph by radar. The inclusion of tire forces during the crash would have increased the interval by a couple miles per hour. Therefore, neglecting ground forces in our analysis gives a conservative result in terms of the impact speed of the striking vehicle.

The analysis just presented was for a perpendicular crash. If the striking vehicle comes in at an angle different than 90°, then the force acting to create the torque must be adjusted by a factor defined as the sine of the angle of incidence. As the angle gets more shallow (i.e., closer to zero), then the solution becomes very sensitive to the angle and the presence of ground forces. It is recommended not to attempt this type of analysis for collisions that are less than 45°.

Exercises

Exercise 14.1. A vehicle is traveling around a 450-ft radius curve at 55 mph.

$\boxed{14.44\,\text{ft/s}^2}$ a.) What is its lateral acceleration?

$\boxed{0.448}$ b.) What is its lateral acceleration factor?

$\boxed{85.2\,\text{lb}}$ c.) What is the lateral force on the driver if he weighs 190 lbs?

Exercise 14.2. An engine develops 250 foot-pounds of torque at 3000 rpm. What is the horsepower developed?

$\boxed{142.8\,\text{hp}}$

Exercise 14.3. If a car weighs 3200 lb, what is its approximate yaw moment of inertia

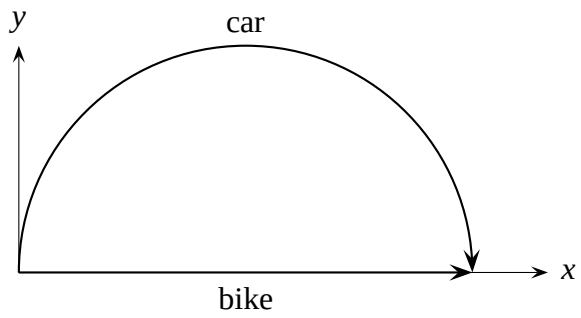
$\boxed{2090}$ a.) in ft-lb-sec^2?

$\boxed{25,080}$ b.) in in-lb-sec^2?

$\boxed{2090}$ c.) in slug-ft^2?

$\boxed{2834}$ d.) in kg-m^2?

Exercise 14.4. A dirt bike racer and a NASCAR driver are at a circular race track. The dirt bike racer challenges the race car driver to see who can get to the opposite end of the track the fastest. The dirt bike will head straight across the diameter and the race car is constrained to the half circle. The top speed of the dirt bike is 90 mph and the top speed of the car is 180 mph. Both vehicles start from a stop at the same point and finish at the same time. The diameter is 1 mile and the race last 60 seconds.[†]



[†]For this exercise we will assume a constant acceleration. In reality, vehicles have their highest acceleration at low speeds and taper off when reaching maximum speeds.

a.) What is the magnitude and direction of the acceleration vector of the car 15 seconds after starting? (Assume a constant acceleration of $4.61\,\text{ft/s}^2$) $4.95\,\text{ft/s}^2 \angle 57^\circ$

b.) Determine the magnitude and direction of the acceleration vector of the car while it is at top speed. $26.37\,\text{ft/s}^2$ toward the cente

Exercise 14.5. A 4400 pound vehicle is going up a 25% slope. Its tires have a rolling radius of 16 inches.

a.) How much torque must be developed at the rear axle for the vehicle to climb the slope? $1455\,\text{ft-lb}$

b.) The vehicle has a 60% front and a 40% rear weight distribution and is equipped with a rear locking differential. What surface friction coefficient is required for the vehicle to climb the slope without tire slip? 0.625

c.) The engine in the vehicle can produce 100 foot-pounds of torque at low speed. What overall gear ratio is required to deliver the necessary torque to the rear wheels? 14.55

d.) The rear axle ratio of the vehicle is 3.73:1, what is the transmission ratio required to go up the hill? 3.9

e.) If the vehicle is in a gear that gives the exact gear ratio, will the vehicle be able to accelerate up the hill? Why? no

Exercise 14.6. A 2600-kg light truck is going up a 20° ramp on the way to becoming airborne. Its brakes are locked. The center of mass of the vehicle is located 2.5 meters in front of the rear axle and 90 cm high.

a.) How much pitching torque is placed on the vehicle when the front wheels clear the ramp? $59,900\,\text{N-m}$

b.) What is the approximate pitch moment of inertia about the center of mass? $200.5\,\text{kg-m}^2$

c.) What is the approximate pitch moment of inertia about the rear tire contact patch? $1855.6\,\text{kg-m}^2$

d.) Assuming a constant torque, how much time will elapse before the vehicle is level? $0.465\,\text{sec}$

e.) Is the constant torque assumption more valid for high speeds or low speeds? Why? high

Exercise 14.7. A vehicle with a moment of inertia of 2500 lb-ft-sec^2 is hit eccentrically. This causes an angular velocity of 13 radians/second.

$211,250$ ft-lb a.) How much kinetic energy is in this spin?

b.) If the vehicle weighs 3500 pounds and translates 10 feet with an overall drag factor of 0.56, how much kinetic energy is dissipated in the translation?

$19,600$ ft-lb

spin c.) Is there more energy associated with the spin or the translation?

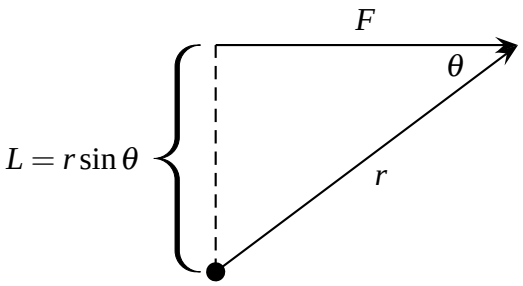


Figure 14.6: The magnitude of torque is the perpendicular distance of the radius multiplied by the magnitude of the force.

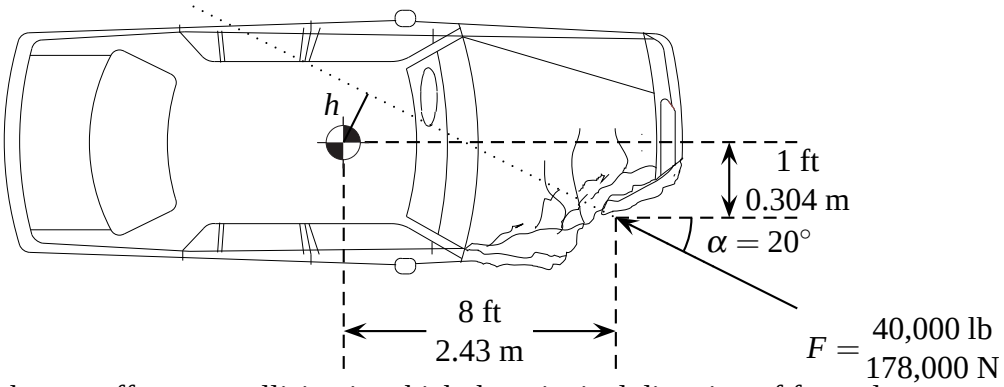


Figure 14.8: A vehicle has an off-center collision in which the principal direction of force does not go through the center of mass.

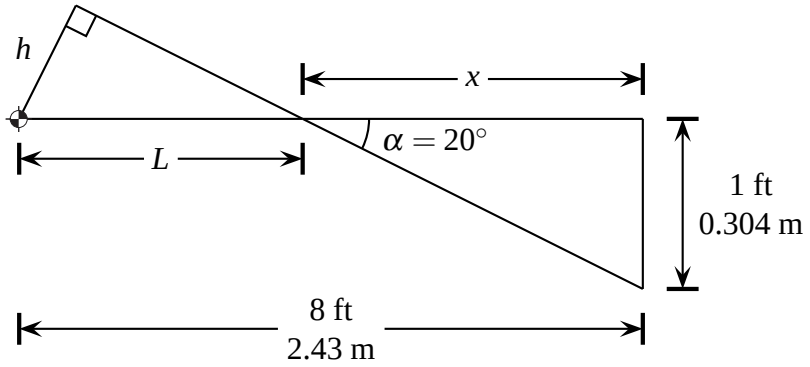


Figure 14.9: Determine the lever arm *h* by analyzing similar triangles.

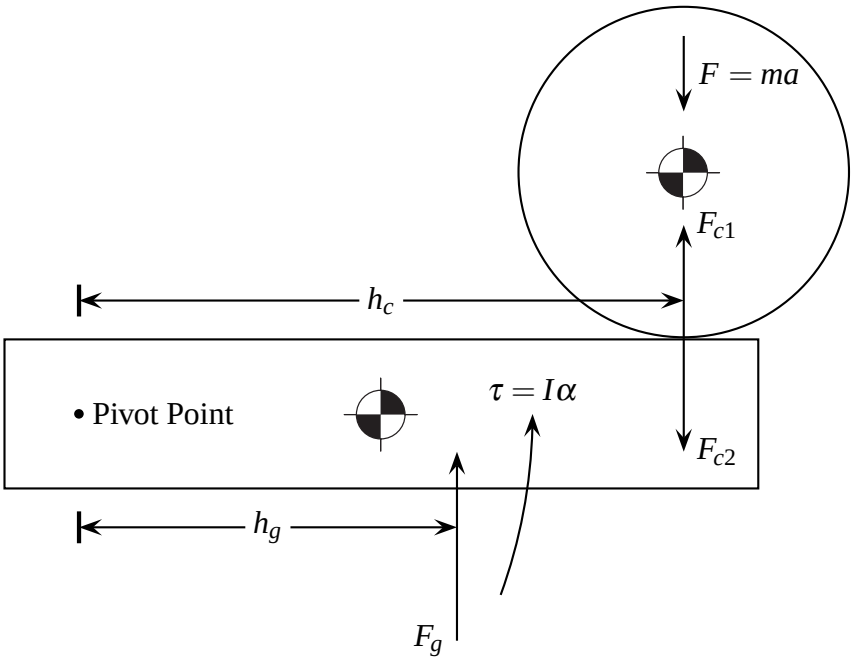


Figure 14.11: A ball strikes a board with an initial velocity and creates a torque about the pivot point. The board is resting on a surface with a drag factor of f. The overall ground force, $F_g = wf$, acts a distance h_g away from the pivot point.

Part III

Laboratory Exercises

15 Laboratory Policies

15.1 Safety

During the first week of classes, every student must read and sign the rules and safety procedures to be eligible to work in the laboratory or shop environment. This sheet will be provided the first lab lesson.

Failure to comply with these rules may result in termination of your privilege to work in the Mechanical Engineering Shops and Laboratories.

In the event of a minor emergency:

Dial extension 5555 for campus security

In the event of a major emergency:

Dial 9-911 for EMS

15.2 Laboratory Schedule

The schedule is randomly chosen to give students an opportunity to work with one another. Labs will meet every week until they are finished, unless otherwise noted. A shedule will be provided.

16 Motor Control and Speed Sensing

16.1 Introduction

Motor speed control for this lab uses a Variable Frequency Drive (VFD) on a common industrial 3-phase motor. The following parts may be used for this lab:

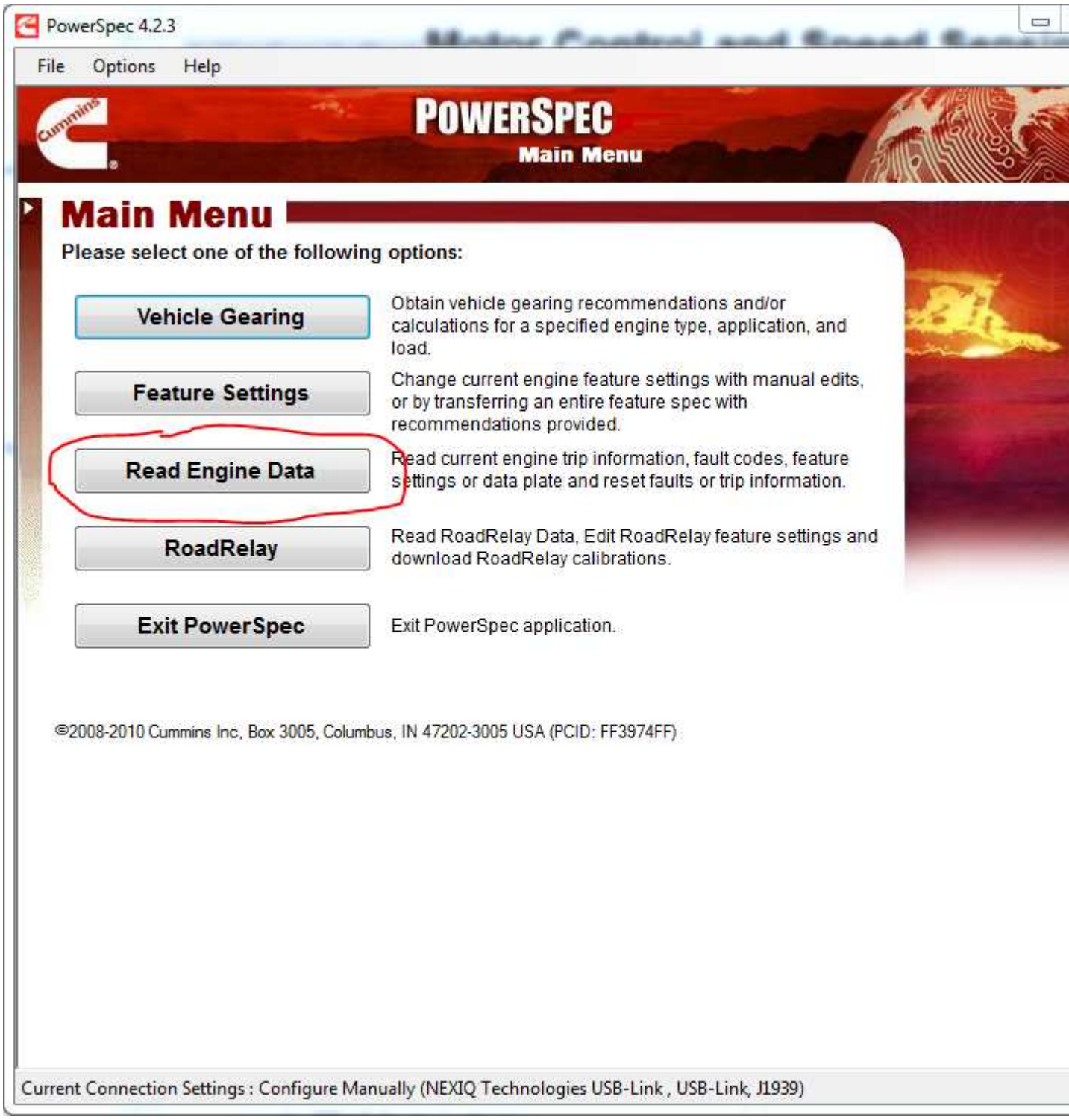
1. 3-phase Electric Motor
2. Variable Frequency Drive
3. Tail-shaft tone ring from a commercial vehicle
4. Tail-shaft speed sensor
5. A Cummins Engine Control Module
6. Vehicle Diagnostics Adapter
7. USB to RS-485 adapter

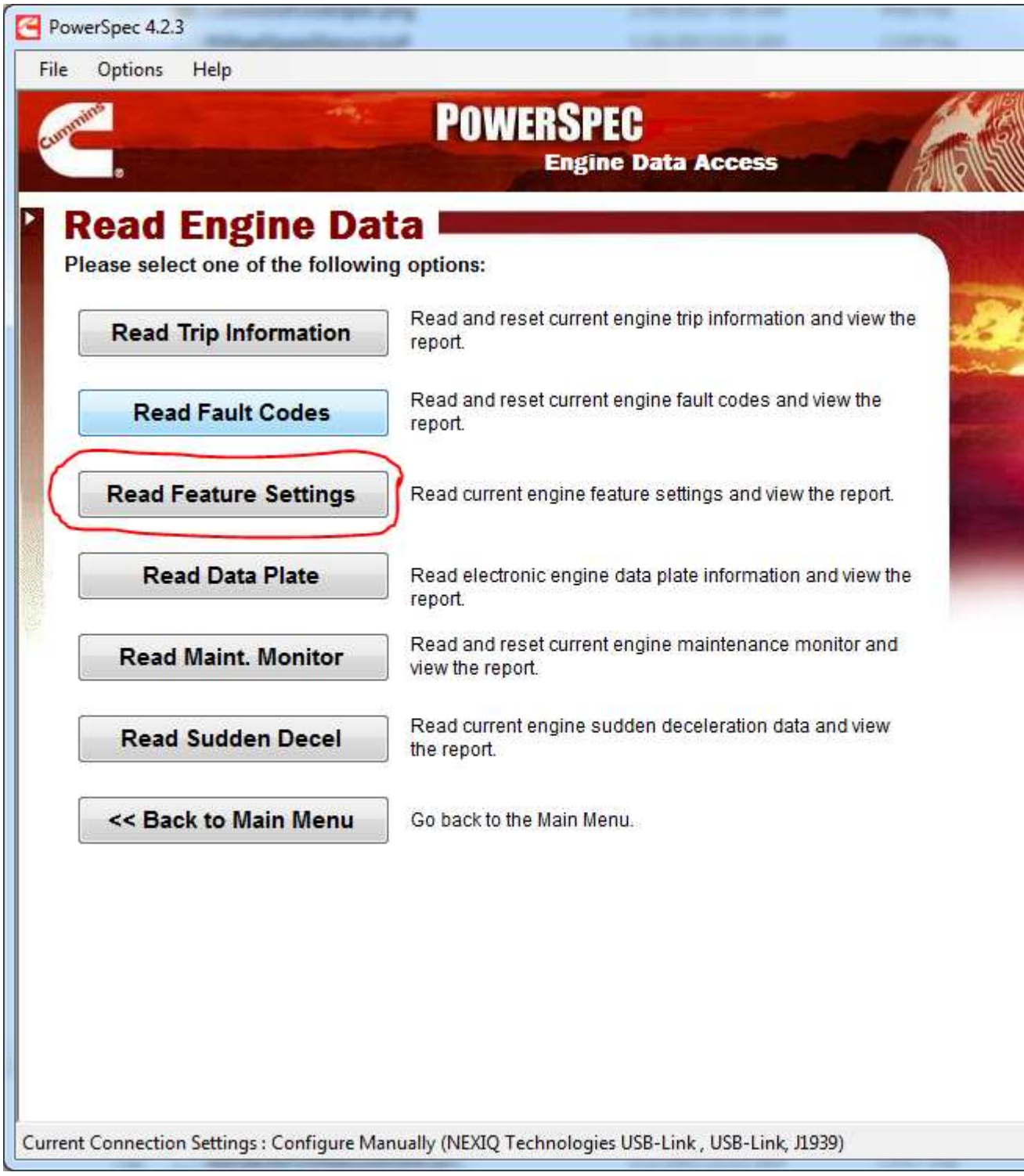
16.2 Assignment

1. Produce a graph that maps frequency from the VFD to Motor RPM. Use at least 10 points and run the motor in both forward and reverse. The rpm must be determined through the vehicle speed signal from the ECM. This speed signal will be J1939 compliant and use the CCVS message described in J1939. You may have to find reference material to aid in your understanding. Some reference material is provided on the S: drive.
2. Create a motor control program such that the vehicle speed changes from 40 mph to 30 mph in 1 second. This should set a Sudden Deceleration Event. Prove that this event was set by turning in a Sudden Deceleration Report produced by Cummins Powerspec that has a unique ECM time value. Be sure to run the motor long enough to give a unique value.

16.3 Proceedure

1. Use Cummins PowerSpec to determine the vehicle setup parameters needed to calculate the drive-shaft RPM from wheel-based vehicle speed. Cummins PowerSpec uses the DPA5 Single Application Driver for the Dearborn Protocol Adapter that may need to be set in the Options menu.

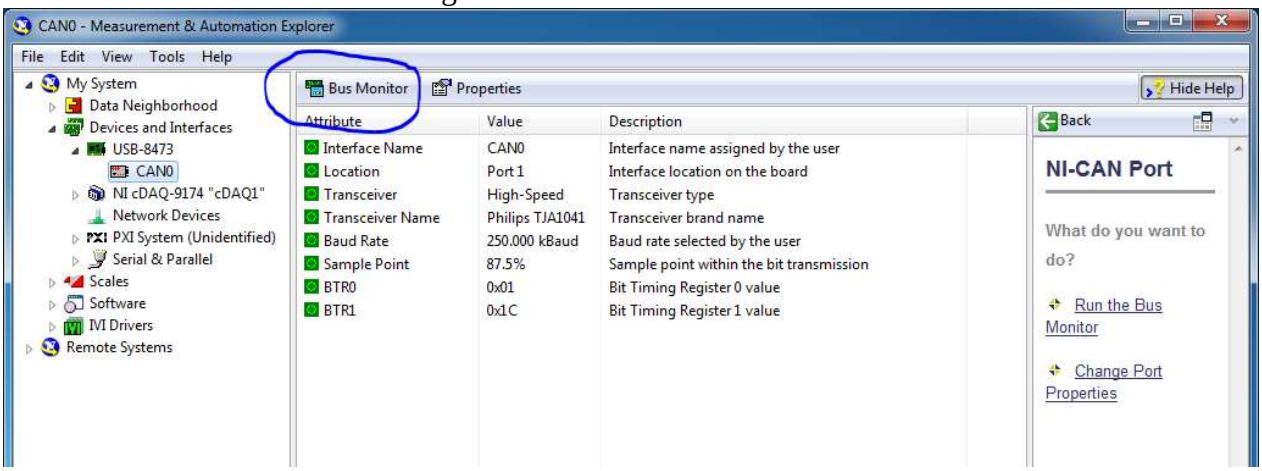

PowerSpec 4.2.3
File
Options
Help
POWERSPEC
Main Menu
Main Menu
Please select one of the following options:
Vehicle Gearing
Obtain vehicle gearing recommendations and/or calculations for a specified engine type, application, and load.
Feature Settings
Change current engine feature settings with manual edits, or by transferring an entire feature spec with recommendations provided.
Read Engine Data
Read current engine trip information, fault codes, feature settings or data plate and reset faults or trip information.
RoadRelay
Read RoadRelay Data, Edit RoadRelay feature settings and download RoadRelay calibrations.
Exit PowerSpec
Exit PowerSpec application.
©2008-2010 Cummins Inc, Box 3005, Columbus, IN 47202-3005 USA (PCID: FF3974FF)
Current Connection Settings : Configure Manually (NEXIQ Technologies USB-Link , USB-Link, J1939)


PowerSpec 4.2.3
File
Options
Help
POWERSPEC
Engine Data Access
Read Engine Data
Please select one of the following options:
Read Trip Information
Read and reset current engine trip information and view the report.
Read Fault Codes
Read and reset current engine fault codes and view the report.
Read Feature Settings
Read current engine feature settings and view the report.
Read Data Plate
Read electronic engine data plate information and view the report.
Read Maint. Monitor
Read and reset current engine maintenance monitor and view the report.
Read Sudden Decel
Read current engine sudden deceleration data and view the report.
<< Back to Main Menu
Go back to the Main Menu.
Current Connection Settings : Configure Manually (NEXIQ Technologies USB-Link , USB-Link, J1939)

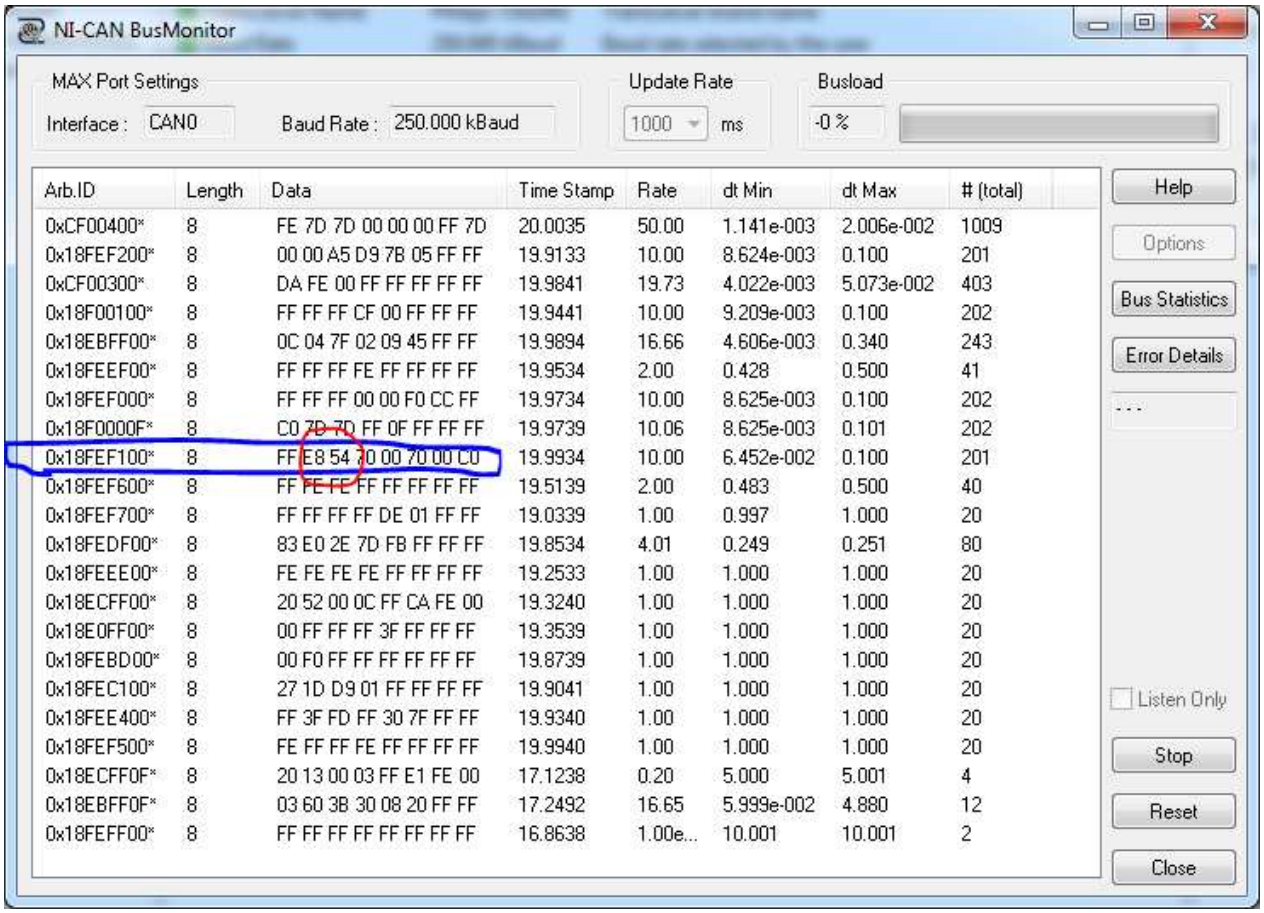
2. Calculate the relationship between Wheel-Based Vehicle Speed in km/h and tailshaft RPM for this setup.

3. Turn on the VFD and set the frequency at different points using the built-in knob.

4. Monitor the J1939 Bus traffic using the NI CAN to USB device in NI MAX software.



5. Use information published by SAE J1939-71 to determine the wheel-based vehicle speed. (Hint: Byte 3 is the integer values of speed in km/h and byte 2 is the decimal values of speed in km/h. This is in reverse byte order and is known as the Intel format.) The first 2 hex digits on the ID are 18 and the last 2 hex digits of the ID are 00.



Arb.ID	Length	Data	Time Stamp	Rate	dt Min	dt Max	# (total)
0xCF00400*	8	FE 7D 7D 00 00 00 FF 7D	20.0035	50.00	1.141e-003	2.006e-002	1009
0x18FEF200*	8	00 00 A5 D9 7B 05 FF FF	19.9133	10.00	8.624e-003	0.100	201
0xCF00300*	8	DA FE 00 FF FF FF FF FF	19.9841	19.73	4.022e-003	5.073e-002	403
0x18F00100*	8	FF FF FF CF 00 FF FF FF	19.9441	10.00	9.209e-003	0.100	202
0x18EBFF00*	8	0C 04 7F 02 09 45 FF FF	19.9894	16.66	4.606e-003	0.340	243
0x18FEEF00*	8	FF FF FF FE FF FF FF FF	19.9534	2.00	0.428	0.500	41
0x18FEF000*	8	FF FF FF 00 00 F0 CC FF	19.9734	10.00	8.625e-003	0.100	202
0x18F0000F*	8	C0 7D 7D FF 0F FF FF FF	19.9739	10.06	8.625e-003	0.101	202
0x18FEF100*	8	FF E8 54 70 00 70 00 C0	19.9934	10.00	6.452e-002	0.100	201
0x18FEF600*	8	FF FF FF FF FF FF FF FF	19.5139	2.00	0.483	0.500	40
0x18FEF700*	8	FF FF FF FF DE 01 FF FF	19.0339	1.00	0.997	1.000	20
0x18FEDF00*	8	83 E0 2E 7D FB FF FF FF	19.8534	4.01	0.249	0.251	80
0x18FEEE00*	8	FE FE FE FE FF FF FF FF	19.2533	1.00	1.000	1.000	20
0x18ECFF00*	8	20 52 00 0C FF CA FE 00	19.3240	1.00	1.000	1.000	20
0x18E0FF00*	8	00 FF FF FF 3F FF FF FF	19.3539	1.00	1.000	1.000	20
0x18FEBD00*	8	00 F0 FF FF FF FF FF FF	19.8739	1.00	1.000	1.000	20
0x18FEC100*	8	27 1D D9 01 FF FF FF FF	19.9041	1.00	1.000	1.000	20
0x18FEE400*	8	FF 3F FD FF 30 7F FF FF	19.9340	1.00	1.000	1.000	20
0x18FEF500*	8	FE FF FF FE FF FF FF FF	19.9940	1.00	1.000	1.000	20
0x18ECFF0F*	8	20 13 00 03 FF E1 FE 00	17.1238	0.20	5.000	5.001	4
0x18EBFF0F*	8	03 60 3B 30 08 20 FF FF	17.2492	16.65	5.999e-002	4.880	12
0x18FEFF00*	8	FF FF FF FF FF FF FF FF	16.8638	1.00e...	10.001	10.001	2

a) Below is a description of the message describing message ID 18FEF100.

PGN 65265 Cruise Control/Vehicle Speed

Transmission Repetition Rate:	100 ms
Data Length:	8
Extended Data Page:	0
Data Page:	0
PDU Format:	254
PDU Specific:	241 PGN Supporting Information:
Default Priority:	6
Parameter Group Number:	65265 (0x00FEF1)

Start Position	Length	Parameter Name	SPN
1.1	2 bits	Two Speed Axle Switch	69
1.3	2 bits	Parking Brake Switch	70
1.5	2 bits	Cruise Control Pause Switch	1633
1.7	2 bits	Park Brake Release Inhibit Request	3807
2-3	2 bytes	Wheel-Based Vehicle Speed	84
4.1	2 bits	Cruise Control Active	595
4.3	2 bits	Cruise Control Enable Switch	596
4.5	2 bits	Brake Switch	597
4.7	2 bits	Clutch Switch	598
5.1	2 bits	Cruise Control Set Switch	599
5.3	2 bits	Cruise Control Coast (Decelerate) Switch	600
5.5	2 bits	Cruise Control Resume Switch	601
5.7	2 bits	Cruise Control Accelerate Switch	602
6	1 byte	Cruise Control Set Speed	86
7.1	5 bits	PTO Governor State	976
7.6	3 bits	Cruise Control States	527
8.1	2 bits	Engine Idle Increment Switch	968
8.3	2 bits	Engine Idle Decrement Switch	967
8.5	2 bits	Engine Test Mode Switch	966
8.7	2 bits	Engine Shutdown Override Switch	1237

b) Below is a description of how to interpret the speed value from the data. (Hint: the programmer's mode in Windows Calculator can convert Hex to Decimal.)

SPN 84 Wheel-Based Vehicle Speed

Speed of the vehicle as calculated from wheel or tailshaft speed.

Data Length:	2 bytes	
Resolution:	1/256 km/h per bit, 0 offset	
Data Range:	0 to 250.996 km/h	Operational Range: same as data range
Type:	Measured	
Supporting Information:		
PGN reference:	65265	

6. Determine Wheel-Based Vehicle speed and transform it to Motor RPM. Create an Excel chart that plots RPM and Wheel Speed (in mph) against Frequency from the VFD. Fit an

appropriate curve through the data.

7. Set up the VFD to accept commands from the computer. Use the PowerFlex user manual on the S: drive for instruction.
8. Using LabVIEW, command the motor to change RPM such that the speed ramps from 40 mph to 30 mph in 1 second. Verify this actually happenes by montoring the voltages from the speed sensor with an analog input module and the Analog Tacho Processing VI.
9. Download the Sudden Deceleration data from ECM using PowerSpec. Repeat for every group member. Ensure different ECM stamps are on the reports.
10. Reset Motor Controller to the default values (Code=P112, Value=1).

16.4 Reporting Requirements

Produce a group website that documents your objective, proceedure and results. Turn in your individual Sudden Deceleration Report from PowerSpec.

17 Determining Inertial Properties

17.1 Problem Statement

Design an experiment to measure the intertial properties of a mountian bike.

17.2 Problem Solving Strategy

Use fundamental engineering problem solving skills to accomplish this task. A problem solving algorithm that may be usful is as follows:

1. Restate and unserstand the problem. You may need to break the problem into a series of smaller problems to solve. Develop metrics to assess success.
2. Assess available resources and constraints.
3. Brainstorm ideas and concepts.
4. Draw useful diagrams and schematics.
5. Develop fundamental relationships between measured variables and quantities of interest based on correct engineering principles.
6. Estimate an anticipated value using some simplifying assumptions.
7. Assess the reliability and uncertainty of the different measurement strategies.
8. Implement and apply the ideas. If finished and acceptable, continue. If not satisfactory, repeat.
9. Effectively communicate your results.

17.3 Reporting Requirements

Turn in a worksheet with instructions on how to conduct the experiment. Also, provide the answers for the worksheet and include all internedusate reasykts,

18 Jumping Impulse

18.1 Objective

This lab has two major objectives: 1) to measure and understand impulse and how it relates to force and change in velocity, and 2) understand how multiple linear regression works for calibrating a force plate.

18.1.1 Calibration Using Multiple Regression

Often real applications require using multiple variables and simple line fitting is insufficient. This lab aims to introduce the student to the concept of multiple linear regression.

18.1.2 Measure Force and Determine Impulse

After completing this lab, students should have first hand experience relating force, impulse and velocity for straight line motion.

18.2 Theory

18.2.1 Determining Jump Height Based on Impulse

Jumping requires a person to exert a force on the ground (or platform) over some time. This force starts with the static weight of the jumper, then reduces as the jumper squats during the counter movement. The legs then begin pushing back on the floor as the jumper accelerates his/her mass upward. This force builds during the jump then falls to zero as the shoes leave the platform.

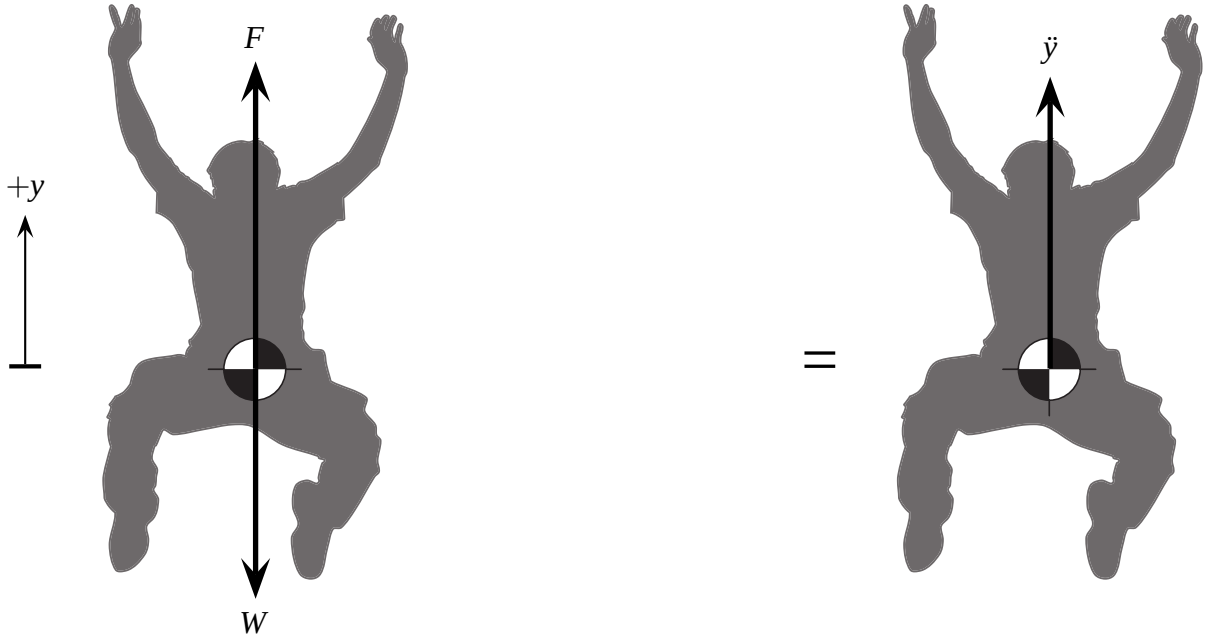


Figure 18.1: The Free body Diagram (FBD) and the Inertial Response Diagram (IRD) as it relates to a jumper.

If we assume the goal of jump height measurements are to determine the height of the center of mass, then the body can be treated as a point mass. Newton's 2nd Lay says:

$$\sum \vec{F} = m\vec{a} \tag{18.1}$$

but for this case, the motion of interest is only in the vertical direction and the forces consist of gravity and contact force. Therefore, the free body diagram along with the inertial response diagram is shown in Fig. 18.1. Newton's Second Law can be rewritten in differential form as

$$\sum \vec{F} = m\frac{dv}{dt} \tag{18.2}$$

where the sum of the forces is $\sum \vec{F} = F - W$ in the y direction. Eq. (18.2) can be integrated over some time:

$$\begin{aligned}
\int_{t_0}^{t} F(t) - W \, dt &= \int_{v(t_0)}^{v(t)} m \, dv \\
\int_{t_0}^{t} F(t) - W \, dt &= m[v(t) - v(t_0)] \\
\int_{t_0}^{t} F(t) - W \, dt &= \frac{W}{g}[v(t) - v(t_0)] \\
\frac{g}{W} \underbrace{\int_{t_0}^{t} F(t) - W \, dt}_{\text{external impulse}} &= \underbrace{v(t) - v(t_0)}_{\text{change in velocity}}
\end{aligned} \tag{18.3}$$

which means a change in velocity can be determined if the external impulse can be measured. If the initial force measurement is taken while the jumper is still, then the static weight W is known. The time before the jump maneuver, when the jumper is still, will be denoted at t_0 and can be any time when the jumper is still. Since the jumper is still, $v(t_0) = 0$.

The time history of force is measured and recorded as a series of forces in time. Since the goal is to determine the integral quantity of impulse, a numerical approximation based on the trapezoidal rule will be used. A formula for determining the impulse from $t = t_0$ to $t = t_n$ is as follows:

$$\int_{t_0}^{t_n} F(t) - W \, dt \approx \frac{\Delta t}{2} \sum_{i=1}^{n} (F_i + F_{i-1} - 2W) \tag{18.4}$$

where there are $n+1$ data points because 0 is used as for the first point. The formula in Eq. (18.4) can be easily implemented in a spreadsheet or computer program. Once the impulse is estimated, the velocity at the end of the impulse can be determined as

$$v(t_n) = \frac{g}{W}\int F\ dt. \tag{18.5}$$

This velocity will determine the take off velocity if the time t_n corresponds to the time when the nor more force is applied to the platform. In other words, the jumper has taken off. Once initial take-off velocity is known, the only force applied to the body is weight. Application of Newton's Law gives

$$\sum F = -W = ma \qquad \text{or} \qquad -W = \frac{W}{g}\ddot{y} \tag{18.6}$$

which leads to a simple 2nd order differential equation: $\ddot{y} = -g$, $y(0) = 0$, and $\dot{y}(0) = v(t_n) = v_o$. Since gravity imposes a constant acceleration, the following kinematic relationship exists:

$$v^2 = v_o^2 - 2gh \tag{18.7}$$

where g is the gravitational constant and h is the height change from the initial position. At the peak height, the velocity is zero, so combining Eqs. (18.7) and (18.5) gives and estimate for jump height:

$$h = \frac{g\left(\int F\ dt\right)^2}{2W^2} \tag{18.8}$$

18.2.2 Jump Height from Hang Time

If the jumper takes off and lands at the same height, then the time traveling up and down are equal. A solution to the constant acceleration differential equation can be obtained through integration and results in a quadratic polynomial:

$$y(t) = y_o + v_o t - \frac{g}{2}t^2 \tag{18.9}$$

Since the times splits between up and down are even, and the time is 1/2 the hang time, the time equation can be written as

$$h = \frac{g}{2}\left(\frac{\text{hang time}}{2}\right)^2. \tag{18.10}$$

Hang-time is easily measured off the time history of force. Keep in mind that the assumption is that the mass center starting height is the same as the landing height. This may be violated if the jumper tuck his/her knees and falls farther than they took off.

18.2.3 Multiple Regression for Calibration

The platform used for this experiment has three load cells at the corners of the triangular structure. Since there are three supports, the sum of the three reaction forces is equal to the applied weight less the weight of the platform W_p. The weight of the platform is unknown and will be included in the measurement as an offset. In equation form, this is written as:

$$F = G_A V_A + G_B V_B + G_C V_C - W_p + \varepsilon \tag{18.11}$$

This is a linear equation and can be written in vector form:

$$\{F\} = \begin{bmatrix} V_A & V_B & V_C & -1 \end{bmatrix} \begin{Bmatrix} G_a \\ G_B \\ G_C \\ W_p \end{Bmatrix} + \{\varepsilon\} \tag{18.12}$$

where the gain vector is constant regardless of the forces and voltages. The residual term ε exists because no measurement system is perfect. There will be slight errors in the process so the linear relationship will not be perfect. The goal of the calibration is to determine the gain constants by minimizing the residual based on multiple measurements. If n measurements are available, then a

series of linear equations can be combined into one equation as follows:

$$\underbrace{\begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix}}_{\mathbb{Y}} = \underbrace{\begin{bmatrix} V_{A,1} & V_{B,1} & V_{C,1} & -1 \\ V_{A,2} & V_{B,2} & V_{C,2} & -1 \\ \vdots & \vdots & \vdots & \vdots \\ V_{A,n} & V_{B,n} & V_{C,n} & -1 \end{bmatrix}}_{\mathbb{X}} \underbrace{\begin{Bmatrix} G_a \\ G_B \\ G_C \\ W_p \end{Bmatrix}}_{\beta} + \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{Bmatrix} \tag{18.13}$$

or

$$\mathbb{Y} = \mathbb{X}\beta + \vec{\varepsilon} \tag{18.14}$$

If n is less than 4, then the system cannot be solved because it is under-determined. If $n = 4$ then the system gives a unique solution, but if $n > 4$, then the system is overdetermined and a solution can be obtained by minimizing the sum of the square of the residuals. This is called the least-squares technique. The sum of squares of the residuals is computed by

$$SSE = \vec{\varepsilon}^T \vec{\varepsilon} = \begin{Bmatrix} \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_n \end{Bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{Bmatrix} \tag{18.15}$$

Equation 18.14 is known as the normal equation and the solution of the overdetermined system that minimizes SSE is

$$\beta = (\mathbb{X}^T \mathbb{X})^{-1} \mathbb{X}^T \mathbb{Y} \tag{18.16}$$

Once β is determined with Eq. (18.16), the gain constants are known as well as the platform weight. Further statistical analysis can be performed on the residuals to ensure that the linear model is the most appropriate, but that is beyond the scope of the course.

18.3 Procedure

This lab requires four parts to complete: 1) Set up the computer DAQ system, 2) Calibrate the force platform, 3) Obtain the data, and 4) analyze the data to determine jump heights. This section will provide the information needed to accomplish these tasks.

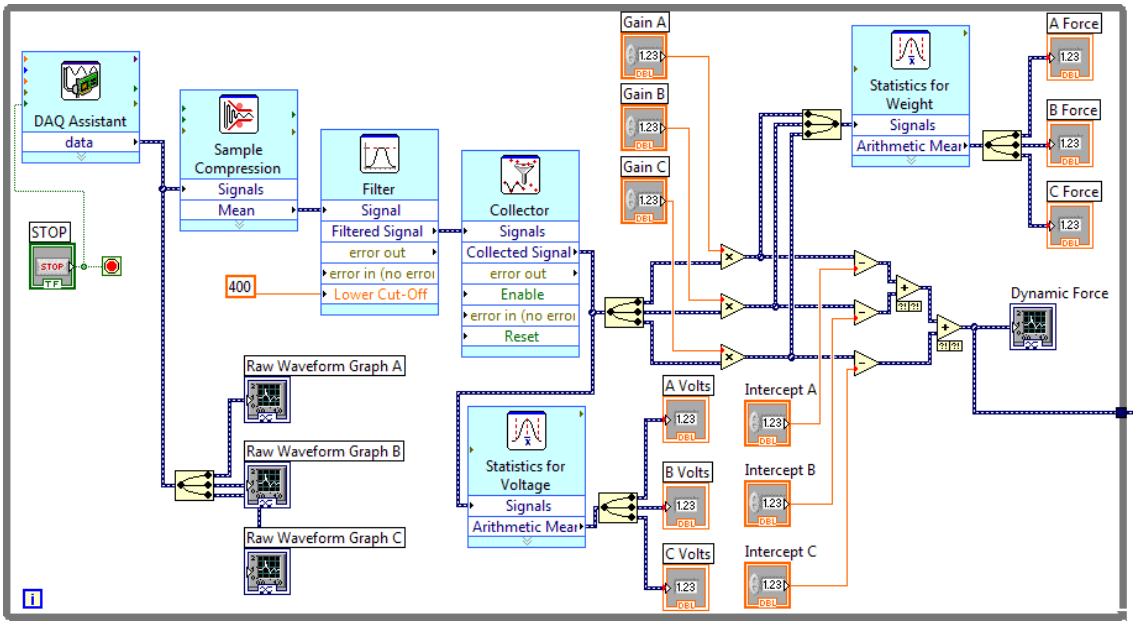


Figure 18.2: LabVIEW Block Diagram for acquiring data from the three independent load cells.

18.3.1 Set Up LabVIEW for Data Acquisition

Open LabVIEW and create a VI capable of measuring the raw voltage from the load cells. The load cells are 2500# shear-beam load cells with a full bridge strain gauge sensor grid. The output on the sensors is 3mV/V at full scale. To use these sensors, ensure they are plugged into the NI 9237 module of the CompactDAQ and the CompactDAQ is on. Once the physical hardware is connected, build a VI similar to the ones shown in Figures 18.2 and 18.3. It is recommended to build and test the DAQ Assistant and Raw Waveform Graphs first.

When starting, use a value of 1 for all the gain constants and a value of 0 for all the intercepts. This will make the outputs simply the raw voltage.

18.3.2 Determine Gain Constants

Once the VI works and displays the average voltage over a period of time, the calibration can commence. Start by obtaining baseline data and record the voltage outputs for no load. Then take a 65-lb dumbbell and place it anywhere on the platform. Record the new voltages. Move the weight to another location (as far away as possible) and repeat recording the voltages. A table for gathering the data may look like the on shown in Table 18.2.

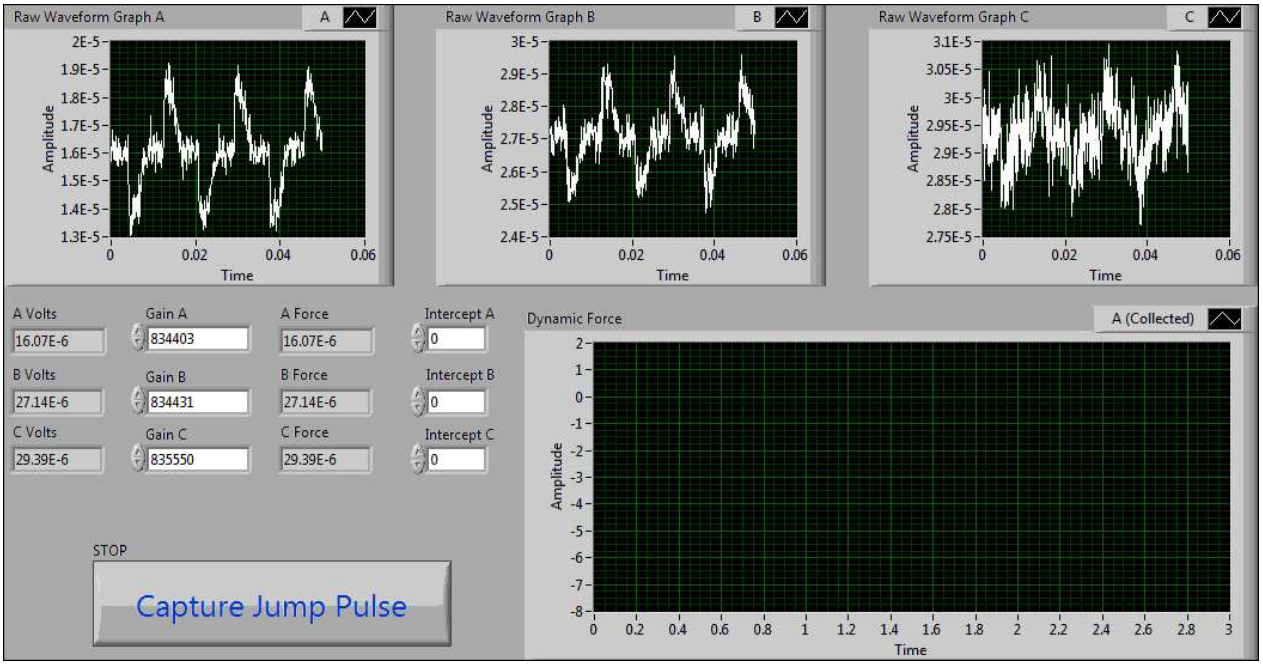


Figure 18.3: LabVIEW Front Panel for acquiring data from the three independent load cells.

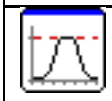
Icon	Tool	Properties
	DAQ Assistant	Sample Rate: 5k Samples to read: 1000 (continuously) Full Bridge with 5V internal excitation
	Sample Compression	Reduction Factor: 5 Reduction Method: Mean
	Filter	Filter Type: Lowpass Cut-Off: 1000 IIR/FIR: Infinite Impulse Response (IIR) Filter Topology: Butterworth Order: 4
	Collector	Number of Points to Collect: 20000

Table 18.1: LabVIEW VI Configuration Properties

Total Weight (lb)	Location 1	Location 2	Voltage A	Voltage B	Voltage C
0	N/A	N/A			
65	middle	N/A			
65	over A	N/A			
65	over B	N/A			
65	over C	N/A			
130	middle	over A			
130	middle	over B			
130	middle	over C			
130	middle	middle			
195	anywhere				
260	anywhere				

Table 18.2: Data collection table for calibration. Please ask for assistance when moving weights if you are uncomfortable with the task.

Use the table to build the $\mathbb{X}$ matrix and $\mathbb{Y}$ vector in Eq. 18.14 and solve for β to get the gains. Next, enter the calculated gain values into the LabVIEW program to get a signal output in pounds.

Finally, each channel needs to be zeroed. Therefore, run the instrument with no weight on the platform. Record the steady output for the individual channels and use those for intercept values in the LabVIEW vi. Be sure that the dynamic with is zero when no objects are on it. The platform should be calibrated at this time.

18.3.3 Measure Impulse and Height

In this section, the process of jumping and recording data are given. This section is optional and not required for the lab. Example data, as shown in Figure 18.4, are available for download from the course website in the form of a Matlab .m file from the following link:

http://personal.utulsa.edu/∼jeremy-daily/ME4024/JumpExample.m

1. Have the jumper mount the platform and remain still. Obtain a static weight.
2. Have the jumper reach as high as possible and touch the slap sticks. Be sure that the maximum reach is near the bottom of the stack of sticks by adjusting the slap-stick height.
3. Perform a couple practice jumps to get the feel of the platform and timing on the data acquisition. Jump using two feet and reach as high as possible on the slap-sticks.
4. Save the time history of force to a data file and import the data into Excel or Matlab for further analysis.

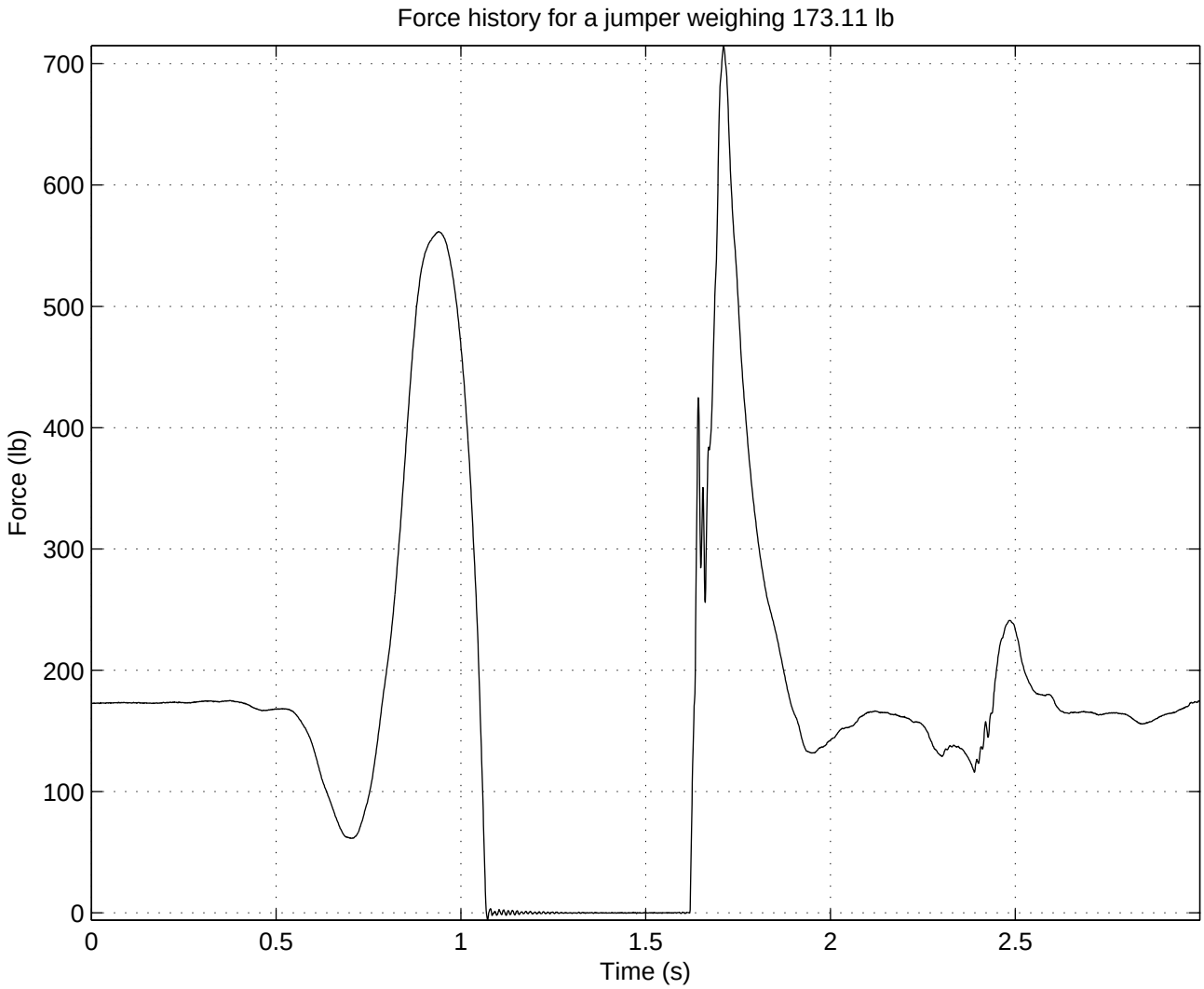


Figure 18.4: Example jump profile data including three phases: take-off impulse, hang-time, and landing impulse.

18.3.4 Data Analysis

There are three different segments to measure jump height from the impulse: the take off impulse, time in the air, and landing impulse. The force vs. time history is capable of providing enough information to perform the analysis. Use the example file or your own file to perform the analysis based on the Theory section of the lab. In Matlab, the `cumtrapz` command can be useful for determining impulse.

18.4 Reporting Requirements

Please include the following in your lab notebook:

1. A print of your LabVIEW Front panel
2. A print of your LabVIEW Block Diagram
3. Raw Data from your calibration procedure in table form
4. Determination of your gain constants
5. A record of the offsets needed to zero the platform
6. Determination of the jump height from the slap-sticks
7. A graph of the jump and landing profile (i.e. the force time history)
8. Determination of the takeoff impulse
9. Determination of the jump height from takeoff impulse
10. Determination of the hang time
11. Calculation of the jump height given hang time
12. A table summarizing the jump height measures from 1) slap sticks, 2) take-off impulse, and 3) hang-time. Also include relative differences assuming the slap-sticks as the standard.
13. Please note the similarities and differences between the measures. Comment on which one you think is most reliable.

19 Balancing Rotating Masses

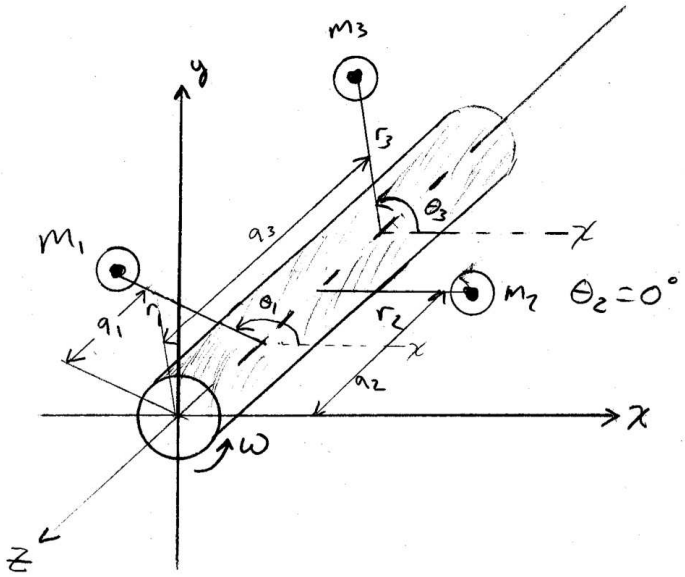
To be conducted the first half of the semester.

19.1 Objective

The purpose of this lab is to balance a rotating shaft. The level of vibration caused by an unbalance will be monitored by an accelerometer.

19.2 Theory

Consider the following schematic of a rotor with three unbalanced masses attached.



The dynamic forces and moments are balanced when the following conditions are met:

$$\sum F_{x,i} = 0$$

$$\sum F_{y,i} = 0$$

$$\sum M_i = 0$$

where the forces due to inertial effects, $|F_i| = m_i r_i \omega^2$. Since gravity and angular velocity are the same for each component on the shaft, they cancel and the following two equations must be satisfied to balance the forces:

$$\sum (wr)_i \cos\theta_i = 0$$

$$\sum (wr)_i \sin\theta_i = 0$$

If these equations are not satisfied, then an unbalance exists. The quantity of wr have units of a moment. A systematic approach to solving this problem is to build a table to set up the system of equations.

Number	Weight, w	Radius, r	Angle, θ	$F_x = wr\cos\theta$	$F_y = wr\sin\theta$
1					
2					
⋮					
N					
Total				0	0

To sum the moments, the dimensions along the shaft must be considered. If the orientation of the shaft is such that the force balance was done in the x-y plane, then the moment balance is done along the z-axis.

$$\sum (wr)_i a_i \cos\theta_i = 0$$

$$\sum (wr)_i a_i \sin\theta_i = 0$$

To assist in formulating the solutions, construct another table.

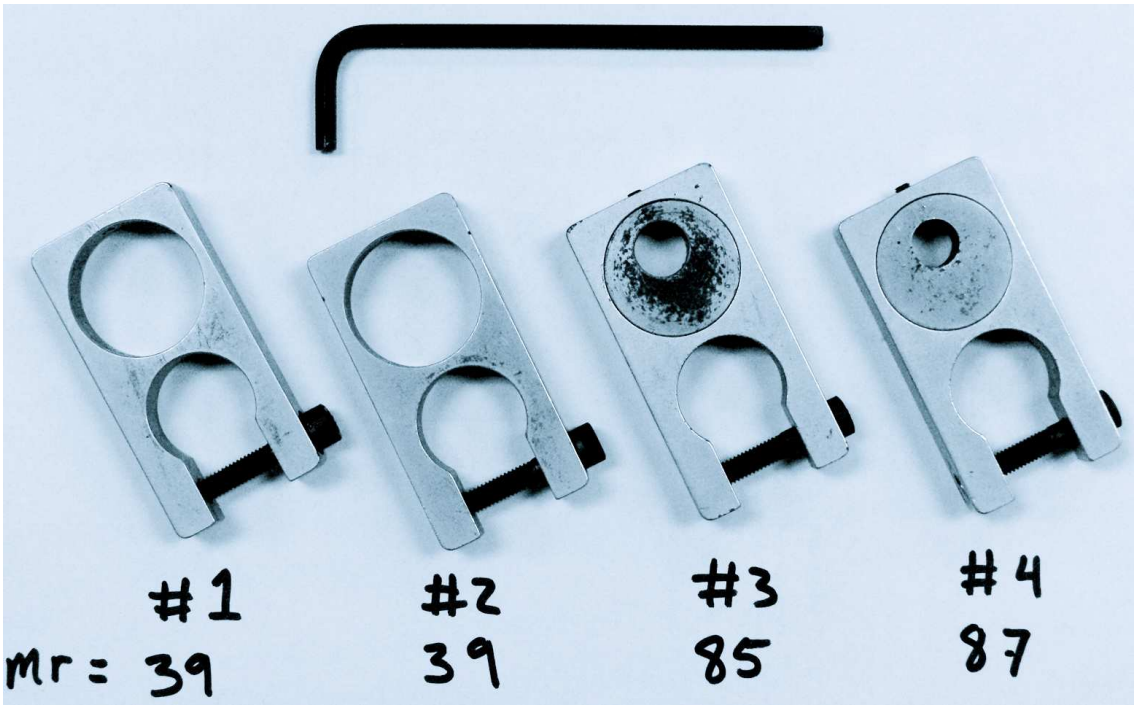
Number	Weight, w	Radius, r	Lever Arm, a	Angle, θ	$M_x = wra\cos\theta$	$M_y = wra\sin\theta$
1						
2						
⋮						
N						
Total					0	0

Note: the two moments are orthogonal and a total moment is calculated as

$$M = \sqrt{M_x^2 + M_y^2}.$$

19.3 Procedure

Find the following parts to use with the Techquipment rotating unbalance machine.



19.3.1 Establish Smooth Baseline

1. Remove all weights using an Allen key.

2. Create a LabVIEW vi to monitor the vibration levels.
3. With the apparatus completely still, zero the accelerometer signal using software.
4. Turn on the motor and observe the smooth operation of the device. Record the Peak-To-Peak and RMS value of the accelerometer signal. (Note: the acceleration signal being monitored is raw voltage and is proportional to acceleration, not the actual acceleration itself.)

19.3.2 Establish Unbalanced Baseline

1. Turn off the motor and add weight #1 (without a center disk) at 0 degrees and 2 cm.
2. Add weight #2 (without a center disk) at 45 degrees and 18 cm.
3. Turn on the motor and observe the vibration. Record the Peak-To-Peak and RMS value of the accelerometer signal.
4. Using the signal acquired from the accelerometer, write some LabVIEW code that will give you the angular velocity of the shaft in RPM. Use a spectrum analysis tool and transform the vibration into the frequency domain. Record this result in your laboratory notebook.

19.3.3 Force Balance

1. Determine the angular position for weight #3 (85 units with center disk) and weight #4 (87 units with center disk) that will balance the rotating inertial forces. To do this, you'll have to solve a simultaneous system of equations. The Solve command in Mathematica may be useful.
2. Place weight #3 at the 12 cm axial position and weight #4 at the 14 cm position.
3. Verify the shaft is in static balance by gently rotating the shaft to different positions as it is horizontal. The shaft should not rotate based on the influence of gravity.
4. Turn on the motor and observe an imbalance caused by the dynamic moment. Record the Peak-To-Peak and RMS value of the accelerometer signal.

19.3.4 Moment Balance

1. Determine the axial positions of weights #3 and #4 that will ensure both force and moment balance.

2. Place weights #3 and #4 at the calculated axial positions.
3. Turn on the motor and observe the system is fully balanced. Peak-To-Peak and RMS value of the accelerometer signal. The the signal from the balanced shaft should be within 10% of the baseline as calculated by

$$\frac{\text{RMS}_{\text{Balanced}} - \text{RMS}_{\text{Baseline}}}{\text{RMS}_{\text{Balanced}}} \times 100\%.$$

19.4 Reporting Requirements

In your lab book, construct (with a straight edge) a

1. force balance table,
2. a moment balance table, and
3. a results table.

The force and moment balance tables are described in the theory section of this hand out and the results table should report the RMS and Peak-To-Peak values of the total acceleration. Also, record the steps taken to write the LabVIEW code used to report the speed of the shaft in RPM.

Have your lab instructor or TA must mark off that the shaft has been balanced.

20 Free Vibration

20.1 Objective

Obtain an understanding of free vibrations by observing and interacting with a vibrating mechanical system. Learn how to determine damping using the log-decrement technique.

20.2 Theory

This section is designed to give you a self guided development of the theory needed for this laboratory exercise.

20.2.1 Differential Equation of Motion

1. Begin with a free body diagram based on the layout shown in Figure 20.1 on page 20-2. The motor is not shown in the figure and more weights may be present. Create your free body diagram on a blank sheet of paper.
2. The free body diagram should have forces from the spring, force from gravity, force from the dashpot, and reaction forces from the fulcrum. The forces are as follows:

   $$F_{dashpot} = c\dot{x}_d$$

   $$F_{spring} = k(x_s - x_{s,o})$$

   where c is the damping coefficient, k is the linear spring stiffness with units of force/length, $\dot{x}_d$ is the velocity of the dash pot link, and x_s is the displacement of the location of the spring attachement. The force from gravity is acting at the center of mass, but the center of mass location is not known. The fulcrum reations are also not known.

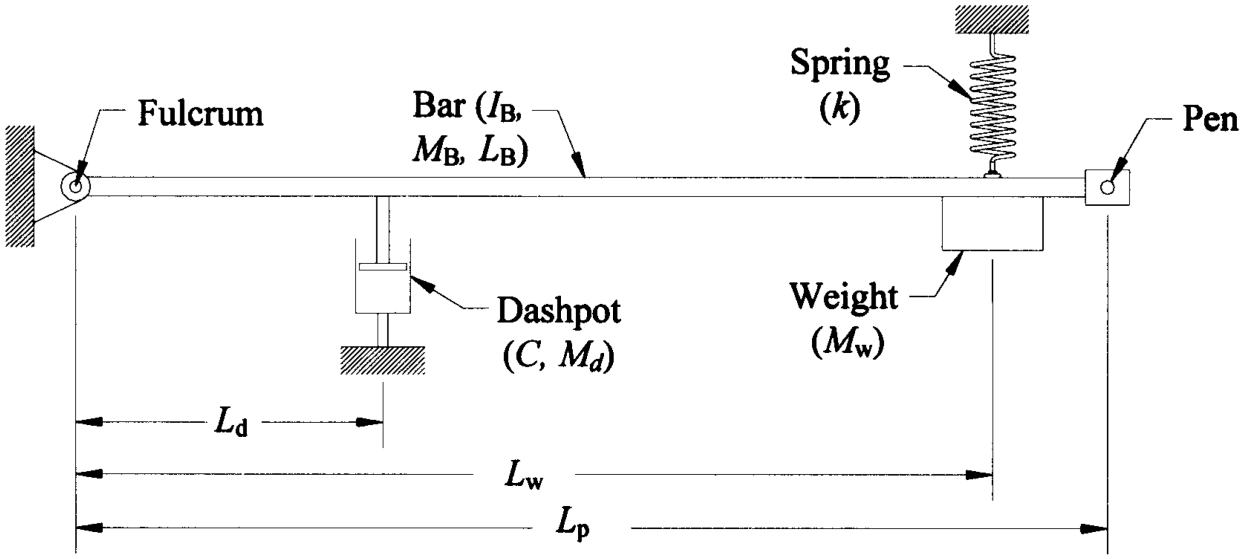


Figure 20.1: Experimental apparatus.

3. With all forces identified, Newton's Second Law can be used to relate accelerations to the sum of forces and moments. The sum of forces along the bar (axial) are neglected. The sum of the forces in the vertical direction are as follows:

$$R_y + c\dot{x}_d - mg + k(x_s - x_{s,o}) = m\ddot{x}$$

Determine the sum of the moments around the fulcurm and write that equation down next to your free body diagram. The moment of inertia is $I_o = mr_{CG}^2 + I_{CG}$ and Newton's Second Law for rotation is

$$\sum M_o = I_o\ddot{\theta}$$

when the subscript o refers to a fixed point.

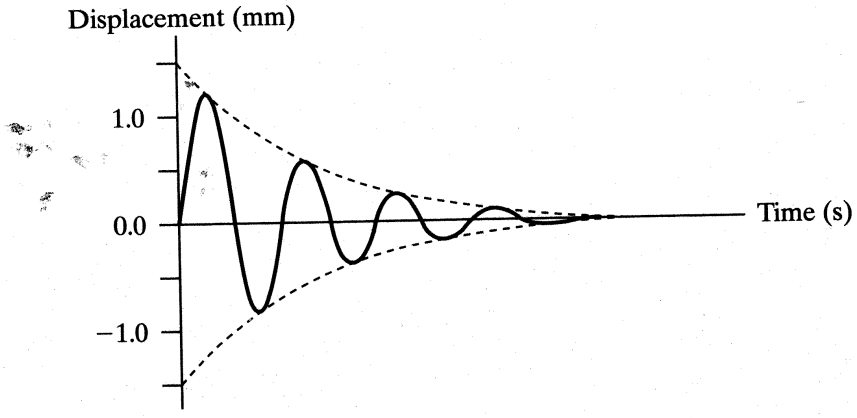
4. The velocities of different points along the bar are different and are related to the angular velocity and the radius. Determine $\dot{x}_d$ in terms of L_d and $\dot{\theta}$.

5. The displacements along the bar can also be determined as a function of angle and radius. Determine x_s as a function of L_s and θ. The same can be done for the location of the pen.

6. Using the moment equation derive a formula that has the following form:

$$A\ddot{\theta} + B\dot{\theta} + C\theta = 0$$

The underdamped free response: $m\ddot{x} + c\dot{x} + kx = 0$ for initial conditions $x(0) = x_0,\ \dot{x}(0) = v_0$



$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi), \quad \omega_d = \omega_n\sqrt{1-\zeta^2}, \quad \omega_n = \sqrt{k/m}, \quad \zeta = \frac{c}{2\sqrt{km}}$$

$$A = \sqrt{\frac{(v_0 + \omega_n\zeta x_0)^2 + (x_0\omega_d)^2}{\omega_d^2}}, \quad \phi = \tan^{-1}\left[\frac{x_0\omega_d}{v_0 + \omega_n\zeta x_0}\right]$$

Figure 20.2: Underdamped Vibrations Solution.

If the neutral position is designated to be $\theta = 0$, then the gravity term drops out of the equation. Determine A, B and C based on measurements L_d, L_s and so forth.

7. Once the homogeneous second order differential equation is obtained, it is necessary to solve it. A solution to this equation can be found in a vibrations book. For example, Inman's Book on Engineering Vibration has the excerpt in the front cover shown in Figure 20.2.

8. The natural frequency (in radians per second) is

$$\omega_n = \sqrt{\frac{\text{stiffness}}{\text{inertia}}}$$

but in the case for the lab, the inertia is a moment of inertia and the stiffness is in units of moment/radian. Write down the solution to the differential equation obtained in Step 6.

20.2.2 Log Decrement

If only the peak values are measured, then the value of the sine function is 1. These points fall along a so-called envelope defined with the following function

$$\text{envelope}=A\exp[-\zeta\omega_n t]$$

If the peaks are measured, an array of values $\{x\}$ can be created that lie on the envelope. Also, these peaks correspond to an array of times

$\{t\}$. Since the peaks are periodic, the ith time $t_i = t_o + T$. If we set $t_o = 0$, then Eq. simplifies to

$$\{x\} = A\exp[-\zeta\omega_n iT]$$

but

$$\omega_n = 2\pi/T$$

so

$$x_i = A\exp[-2\pi\zeta i]$$

Taking the natural logarithm of each side gives

$$\ln x_i = \ln A - \underbrace{2\pi\zeta}_{m} i$$

where m is the slope of the line, $i = 1,2,...n$, and $x_1$is the amplitude of the first peak, x_2 is the amplitude of the second peak and so forth. This curve fitting method can be used to determine the damping ratio ζ.

20.2.3 Indirect Weighing

Sometimes vibrating structures are not conducive to direct weighing methods (i.e. using a scale). In this case, a change in the natural frequency upon addition of additional mass can yield results. The procedure for the case of no damping is as follows:

1. Measure the natural frequency of the system.

$$\omega_{n,1} = \sqrt{\frac{k}{I}} \text{ rad/s}$$

2. Add a known mass, and remeasure the natural frequency.

$$\omega_{n,2} = \sqrt{\frac{k}{I + I_{add}}} \text{ rad/s}$$

3. Use these two equations to solve for I:

$$\omega_{n,1}^2 I = k$$

$$\omega_{n,2}^2 = (I + I_{add}) = k$$

so

$$I = \frac{I_{add}}{\left(\frac{\omega_{n,1}^2}{\omega_{n,2}^2} - 1\right)}$$

Notice that the stiffness is not needed for this method. This moment of inertia can be compared to the moment of inertia calculated from natural frequency and know spring stiffness.

20.3 Procedure

1. Determine the stiffness of the spring by placing weights on a hangar and measuring deflection as shown in Figure 20.3 on page 20-6. Use at least eight points and perform a linear curve fit to obtain stiffness. Give your final values in units of N-m/rad.
2. Remove the weight hangar and all the weights and the dashpot. Be sure not to disassemble the dashpot such that it leaks or gets messy. In other words, remove the top screw above the

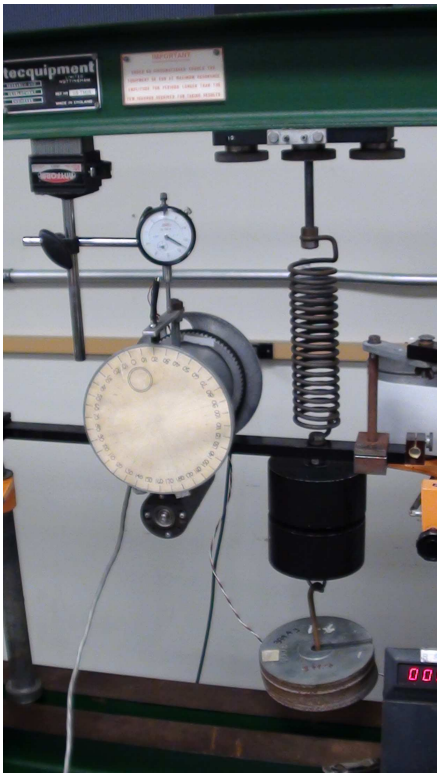


Figure 20.3: Photo showing the determination of the spring stiffness.

bar.

3. With the dashpot detached, cound the number of oscillations for a certain amount of time (like 30 seconds).
4. Add the entire stack of weights (a known mass) to the hangar and repeat the frequency measurement. This will provide a method to determine the moment of inertia.
5. Locate the dashpot 20 cm from the fixed end.
6. Set up the pen to create a trace near the center of the paper. Ensure the paper weight is attached and ensure the paper moves with the drum.
7. Measure the response of an initial displacement using the strip chart.
8. Locate the dashpot 30 cm from the fixed end.
9. Measure the response of an initial displacement using the strip chart.
10. Locate the dashpot 40 cm from the fixed end.
11. Measure the response of an initial displacement using the strip chart.
12. Based on the three strip charts, determine the amplitude of the peaks. Fill in the amplitudes in the table below:

Peak	1	2	3	4	5	6	7	8	9	10	11	12
20 cm												
30 cm												
40 cm												

13. Based on these peaks, fit an exponential curve through the values and determine the damping ratio.
14. Determine the damping coefficient for dashpot by adjusting the damping ratio to account for the location of the dashpot.
15. Determine the moment of inertia from the stiffness and the natural frequency.
16. Determine the moment of inertia from the natural frequency from a baseline system and the natural frequency with added mass.

20.4 Reporting Requirements

Please submit a report according to SAE guidelines describing the lab.

21 Forced Vibration

To be done in the second half of the semester

21.1 Objective

Obtain an understanding of free vibrations by observing and interacting with a vibrating mechanical system. Learn how to determine damping using the log-decrement technique.

21.2 Theory

In the space below, draw a freebody diagram of the vibration apparatus:

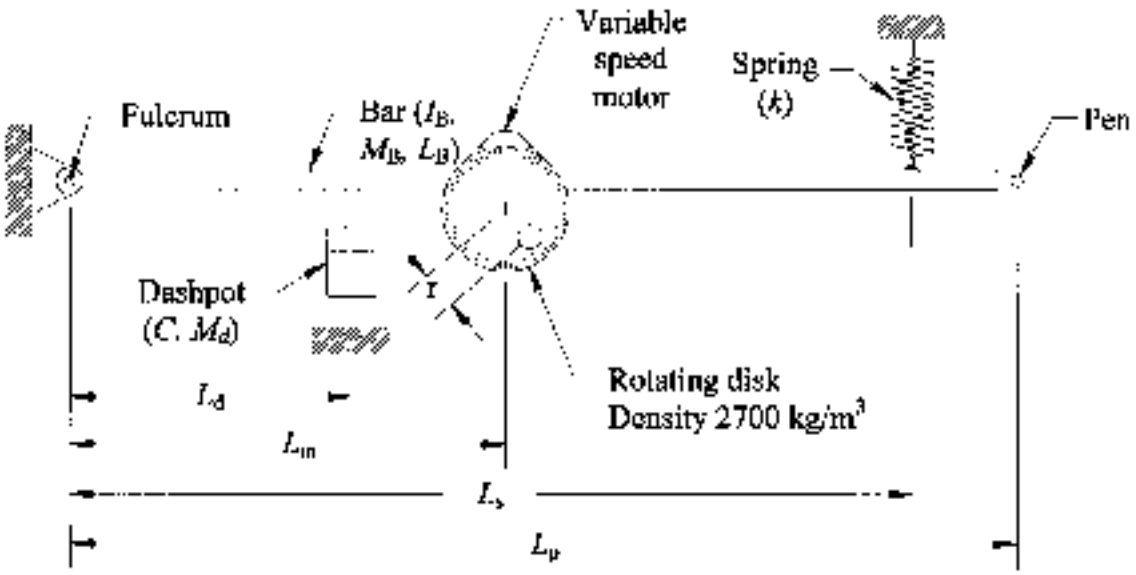


Figure 21.1: Lab apparatus with rotaing unmalanced motor

Derive the equation of motion by summing the moments about the fulcrum. The hole in the plate acts to apply an unbalanced inertial force to the bar. The force is opposite the centripetal force that is accelerating the unbalance toward the center of rotation. Therefore, the forcing function applied at the center of the motor mount is

$$F(t) = me\omega^2 \sin(\omega t)$$

where ωt is the angle of the motor, ω is the angular velocity of the motor (in rad/s), m is the mass of the unbalance, and e is the eccentricity of the unbalanced mass.

The differenatial equation for this system has the form

$$I_o\ddot{\theta} + C\dot{\theta} + K\theta = mel\omega^2 \sin(\omega t)$$

where you need to determine the values of C and K based on your free body diagram, current measurements, and the free vibrations lab.

$C =$

$K =$

Solving the differential equation requires two parts: homogeneous and particular. The homogenous part was solved in the free vibrations lab and it decays to zero as time moves on. The steady-state or particular solution is obtained by assuming a particular solution of the form

$$\theta_p(t) = X\sin(\omega t - \phi)$$

where X needs to be determined and ϕ is the phase angle. Let us define a frequency ratio as

$$r = \frac{\omega}{\omega_n}$$

where

$$\omega_n = \sqrt{\frac{K}{I_o}}$$

is the natural frequency of the vibrating system. The solution for X and ϕ are as follows:

$$X = \frac{mel}{I_o}\frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \tag{21.1}$$

and

$$\phi = \tan^{-1}\frac{2\zeta r}{1-r^2} \tag{21.2}$$

A plot of the transfer funtion is shown in Fig.

21.3 Procedure

A handout will be provided on-line for this lab.

21.4 Reporting Requirements

A full formal report comparing the different methods for determining the damping ratio of the dashpot.

1. This lab requires a report. The report will depend on results from both the forced and free vibration labs. The following questions should be addressed in the report.

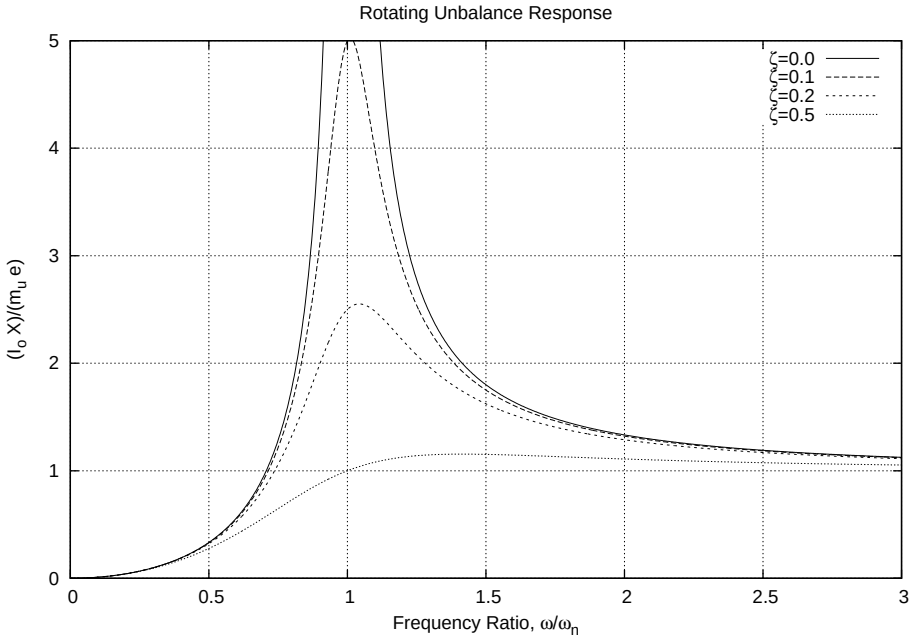


Figure 21.2: Plot of the amplitude function from Eq. 21.1.

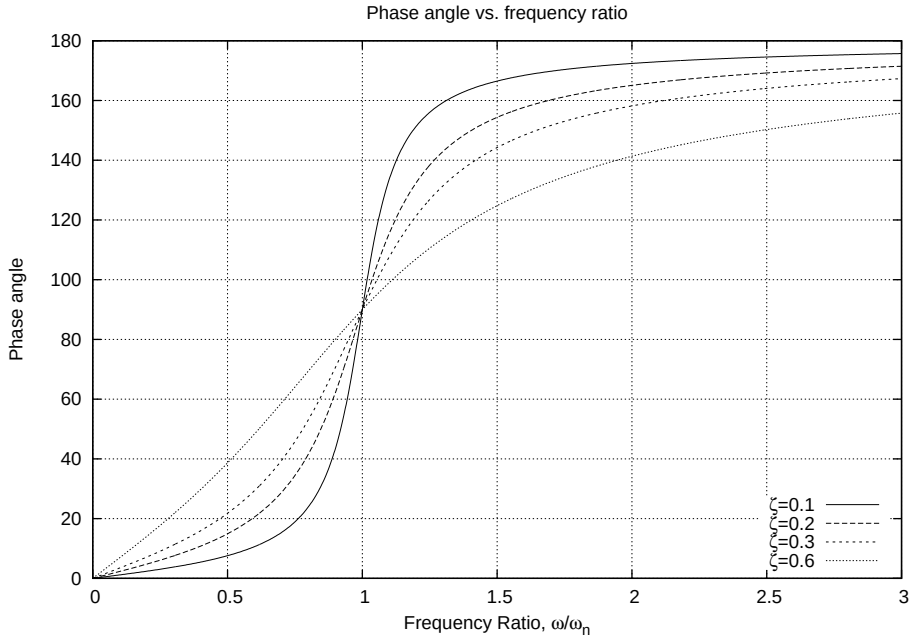


Figure 21.3: Plot of the phase function from Eq. 21.2.

2. What is the moment of inertia of the assembly in the free vibration lab?
3. What is the linear spring stiffness in the free vibration lab and in the forced vibration lab?
4. What is the damping coefficient of the dashpot based on the log-decrement technique from the free vibration lab?
5. Provide an estimate and justification for the natural frequency of the system in both the forced and the free vibration lab.
6. What is the damping coefficient of the dashpot based on the magnitude of the forced response?
7. Using the average of the measured damping coefficients, plot the predicted phase vs angular velocity on the same graph as the measured phase. Comment on the results.

22 Small Engine Rebuild

To be conducted in the first half of the semester in the McElroy Prototyping Laboratory (L112).

22.1 Objective

Reciprocating machinery is ubiquitous in machinery and understanding the details of their construction is important in that it brings relevance to the modeling of the dynamics of machinery. This lab exercise is to show the student how the internal components of an engine are assembled. Also, the internal engine components will be measured and engineering values assigned.

22.2 Tasks

Each lab group will be required to disassemble, measure, and rebuild a single cylinder, four stroke, internal combustion engine. These engines are Briggs and Stratton 5hp vertical shaft engines that may be used on a lawn mower. The engine must run at the completion of the lab. Engines can only be started outside on the loading dock. The following measurements or quantities must be obtained to complete the lab. Many of these measurements will require some calculation and further research.

1. Displacement in cubic inches and cubic centimeters
2. Cylinder head volume (in cubic inches)
3. Compression ratio
4. Stroke (in.)
5. Crank to connecting rod journal clearance (in inches)
6. Valve lift for both intake and exhaust (in inches)

7. Spark gap (in.)
8. Ring drag (lb)
9. Mass of the connecting rod (lb-sec^2/in)
10. Moment of inertia esimate of the connecting rod about the wrist pin
11. Center of mass location of the connecting rod
12. Journal Clearance between the connecting rod and the crank shaft
13. Mass of the reciprocating parts (lb-sec^2/in)
14. Mass of the Crank Shaft (lb-sec^2/in)
15. Moment of inertia estimate of the crank shaft
16. Valve lash (in inches)
17. Output shaft size and key way size
18. Flywheel mass (lb-sec^2/in)
19. Estimate of the flywheel moment of inertia
20. Overall cam lift

The lab will be graded in two parts: 60 points for the above measurements and 40 points if the engine starts after reassembling it.

22.3 Dissassembly

Below is a stepwise list of the disassembly process:

1. Drain the oil via the drain plug located on the bottom of the engine case. Catch the drained oil in the specified pan.

2. Drain the fuel from the fuel tank by removing the hose from the fuel tank itself. Catch this fuel in the specified container.
3. Remove the spark plug and measure the gap using a feeler gauge set. Once complete set the plug aside for later reassembly.

4. Remove the outer half of the air cleaner.

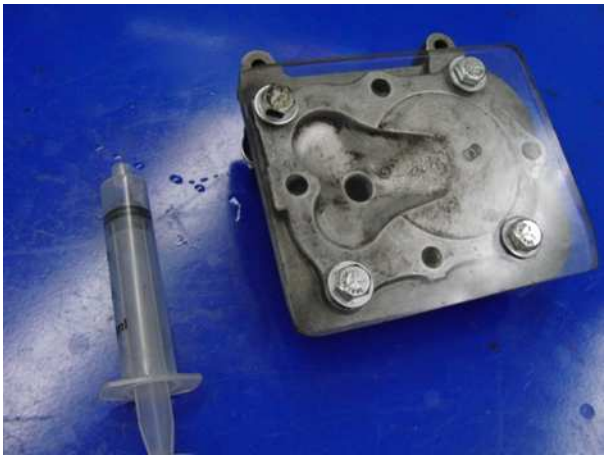
5. Remove the air filter. Remove the air cleaner base from the carb assembly. Be sure to include the vent hose that is attached to the back of the air cleaner base.
6. Remove the fuel tank assembly by removing the small bolts located around the ring near the pull cord, and then the one bolt underneath the tank attached the block. Do not lose the fuel tank spacer that comes off with this bolt.
7. Remove the muffler.

8. Remove the pull cord assembly and the basket. This is accomplished by removing the four outer most bolts– two at each end.
9. Remove the carburator assembly from the engine by unbolting the 2 bolts that attach the intake manifold to the cylinder. Then remove the 2 bolts that attach the carburator assembly to the block.

10. Remove the two bolts holding the brake assembly to the block ear. Carefully remove the wire attached to the brake assembly that grounds the coil.

11. Remove the coil. Note: The air gap required for proper coil operation is 0.010”, which is about the thickness of a playing card or note card.
12. Remove the Cylinder head taking extreme care not to damage the head gasket. Once removed measure the cylinder head volume using the supplied fixture and syringe.

    Note: 1mL = 1cc.



13. Remove the large nut holding the flywheel down.

14. Remove the flywheel using the pulley puller. Once removed begin measurements for the required calculations.
15. Measure the bore of the cylinder.
16. 16. Measure the depth of the cylinder with the piston all the way at the bottom of its cycle. This measurement is the stroke.

    a) Calculate the displacement in cubic inches and cubic centimeters. Note: 1 in. = 2.54 cm

b) Calculate the compression ratio.

17. Measure the valve lift using a dial indicator mounted on a magnetic base.

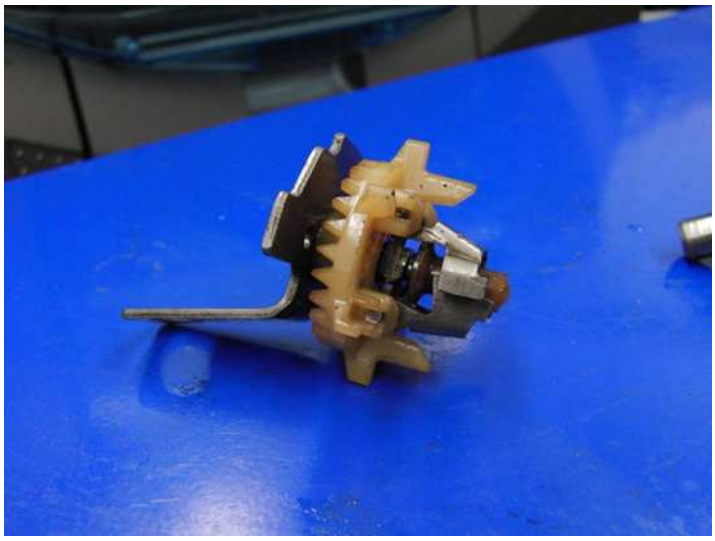
18. Remove the valve cover. Taking care not to damage the gasket.

19. Measure the valve lash using the feeler gauges.

20. Remove the valves, springs and retainers taking care not to mix the parts.

21. Remove the side case cover from the block. Taking care not to damage the gasket.

22. Remove the governor idle assembly



23. Remove the camshaft and tappets.

24. Remove the two bolts attaching the connecting rod to the crankshaft.

25. Push the piston assembly up in the cylinder and out of the way for crankshaft removal.

26. Remove the crankshaft. Once removed begin measurements.

27. Measure drag by placing lab weights on the cylinder until it moves.

28. Once complete, remove the piston and rod assembly. Remove the wrist pin from the piston and separate the connecting rod from the piston.

29. Measure the overall volume of the cylinder.
30. Weigh the reciprocating assembly.
31. Measure the output shaft diameter and keyway size.

32. Determine the center of mass of the connecting rod.
33. Measure the journal clearance bwrween the crankshaft and the connecting rod.
34. Obtain any and all other related measurements including weights required to produce accurate estimates.

35. Clean and inspect all the components.

Reassembly is the reverse of the disassembly operation.

22.4 Reporting Sheet for Engine Rebuild Lab

Please fill in the following table and turn in the sheet to Dr. Daily.

Reporting Sheet for Engine Rebuild Lab	
Name	
Group Members	
Date:	
Time Started:	
Time Finished:	

Measurements with units

Displacement in cubic inches and cubic centimeters	
Cylinder head volume	
Compression ratio	
Stroke	
Crank to connecting rod journal clearance	
Valve lift for both intake and exhaust	
Spark gap	
Ring drag	
Mass of the connecting rod	
Moment of inertia esimate of the connecting rod about the wrist pin	
Center of mass location of the connecting rod	
Journal Clearance between the connecting rod and the crank shaft	
Mass of the reciprocating parts	
Mass of the Crank Shaft	
Moment of inertia estimate of the crank shaft	
Valve lash	
Output shaft and keyway size	
Flywheel mass	
Flywheel moment of inertia	
Overall cam lift	
Did the engine run?	Yes____ or No_____
Signature of lab instructor:	

23 Crash Analysis and Deposition Exercise

23.1 Objective

The objective of this exercise is to evaluate information and determine expert opinions based on dynamics concepts to determine who was at fault in a traffic crash. The opinions will be Provide testimony to a automobile crash.

The purpose of a deposition is for attorneys to obtain sworn testimony from witnesses in an attempt to understand the opinions that will be rendered in court. While the format of a deposition may vary, the following questions and themes are generally followed:

1. When did you become involved with the case?
2. What were you asked to do?
3. What did you actually do?
4. What are your opinions?
5. What is the factual basis for your opinions?
6. Are there any other thoughts or opinions that you would present in court?

For this particular case you will be required to present an enumerated list of your opinions and be able to justify them in a legal proceeding. If you are not deposed, then you will be required to fill out interrogatories to address the same crash.

23.2 The Crash Scenario

On the morning of Tuesday, October 26, 2010 Julie Smith was driving her Buick Century south on Merrily Way while taking three-year old daughter, Kate, to day care. At the same time, Joseph

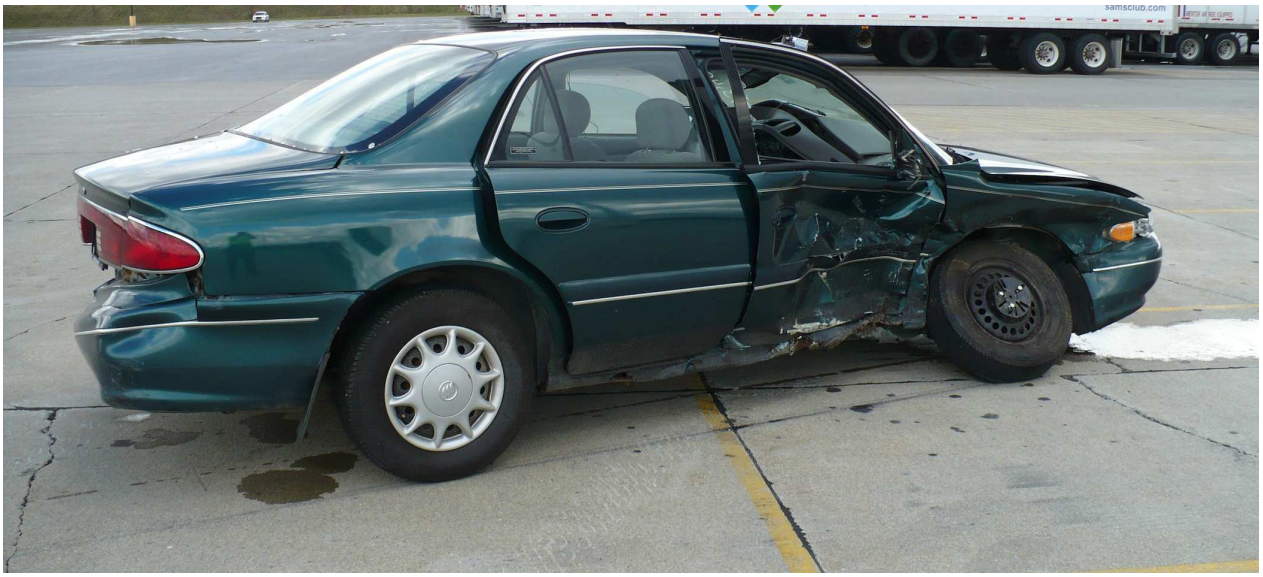


Figure 23.1: Damage to the Buick

Olson was driving westbound on Elm Street in his Hyundai Accent to his first class at the local community college. The morning was crisp and the leaves were starting to turn. It was about 9:00 am and the sun was warming the day in a cloudless sky. It was just like any other normal day. However, both vehicles tried to occupy the same section of pavement at the same time.

The collision was severe enough to put all occupants in the hospital. Ms. Smith suffered a broken arm and concussion. Little Kate had a piece of glass or debris sever her ear. They were both wearing their seat belts. Mr. Olson did not fare as well. His injuries were severe with a compound femur fracture, broken wrist and severe lacerations on the head and neck. Evidence at the scene showed he wasn't wearing his seat belt. After 1 week in the hospital, Mr. Olson passed away. The police obtained a warrant for cell phone records and discovered that Mr. Smith has sent 5 text messages and placed 1 phone call during the morning of the incident. The last text message was placed at 8:57:34 AM and the first 911 call for the crash was at 8:58:45 AM.

An official police report was prepared from the incident that contained information with drawings, dimensions, and weights. The intersection was controlled with a two-way stop sign. The speed limit south bound on Merrily Way was 25 mph and the speed limit on Elm Street was 35 mph. Merrily Way had the stop sign. The stop sign was 25 feet behind the front axle of the Buick when the collision occurred. The conclusion of the police report is to charge Ms. Smith for Vehicular Manslaughter as they claimed she did not stop at the stop sign and her gross negligence lead to the death of Mr. Olson. The law in Oklahoma (Title 47, Chapter 11, Article 9 [ğ 11-903]) for Negligent Homicide states:

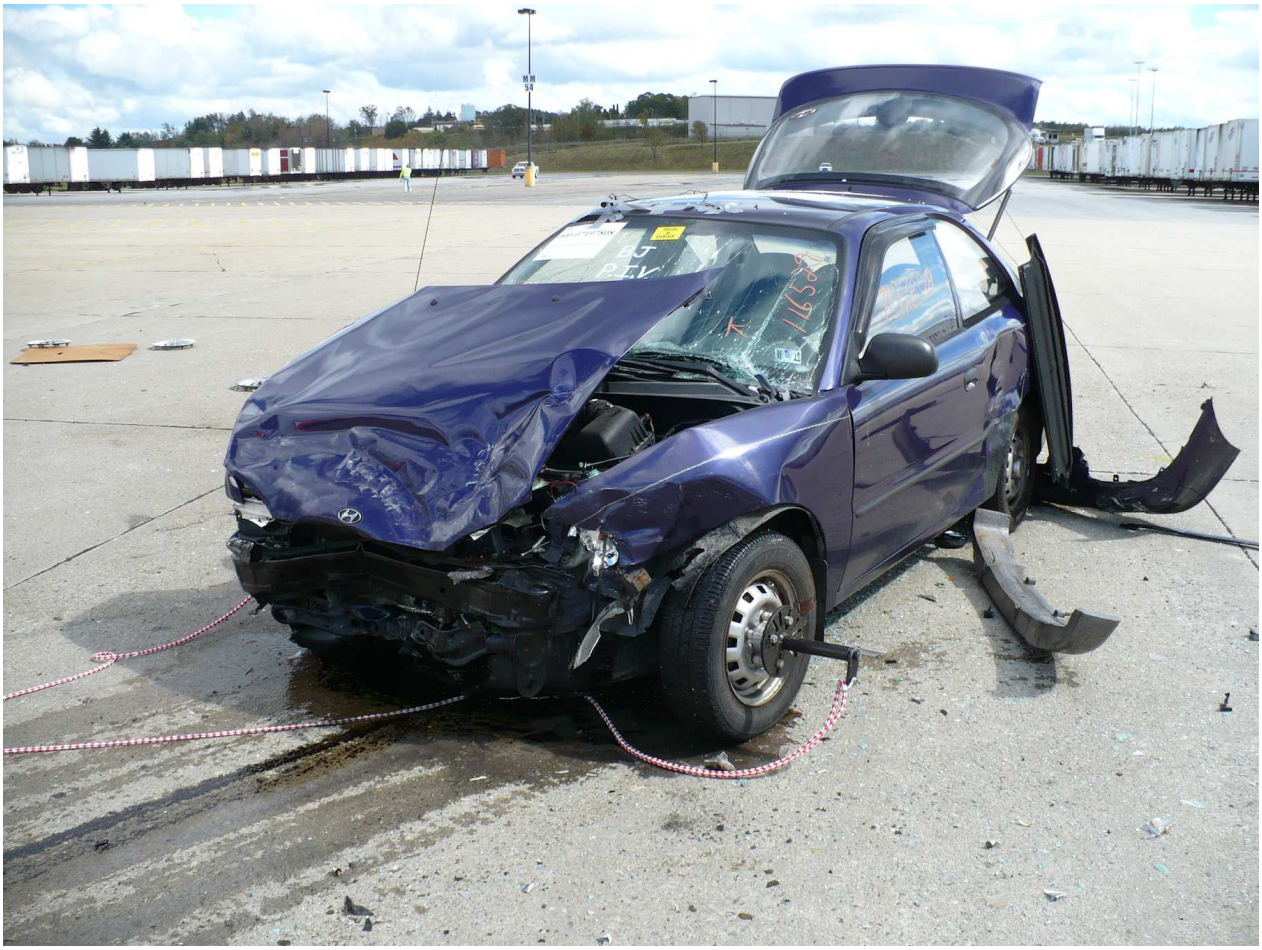


Figure 23.2: Damage to the Hyundai

Figure 23.3: Overhead photo of the crash scene.

> When the death of any person ensues within one (1) year as a proximate result of injury received by the driving of any vehicle by any person in reckless disregard of the safety of others, the person so operating such vehicle shall be guilty of negligent homicide.

Ms. Smith gave a statement to the police that she looked both ways and didn't see anyone. She drives this route frequently and did not remember if she came to a complete stop. She remembered running late that morning and was fearful of losing her job because she was a single mother.

23.3 Supporting Documents

23.3.1 Computer Generated Drawings

The next page has the computer drawing from which to measure distances.

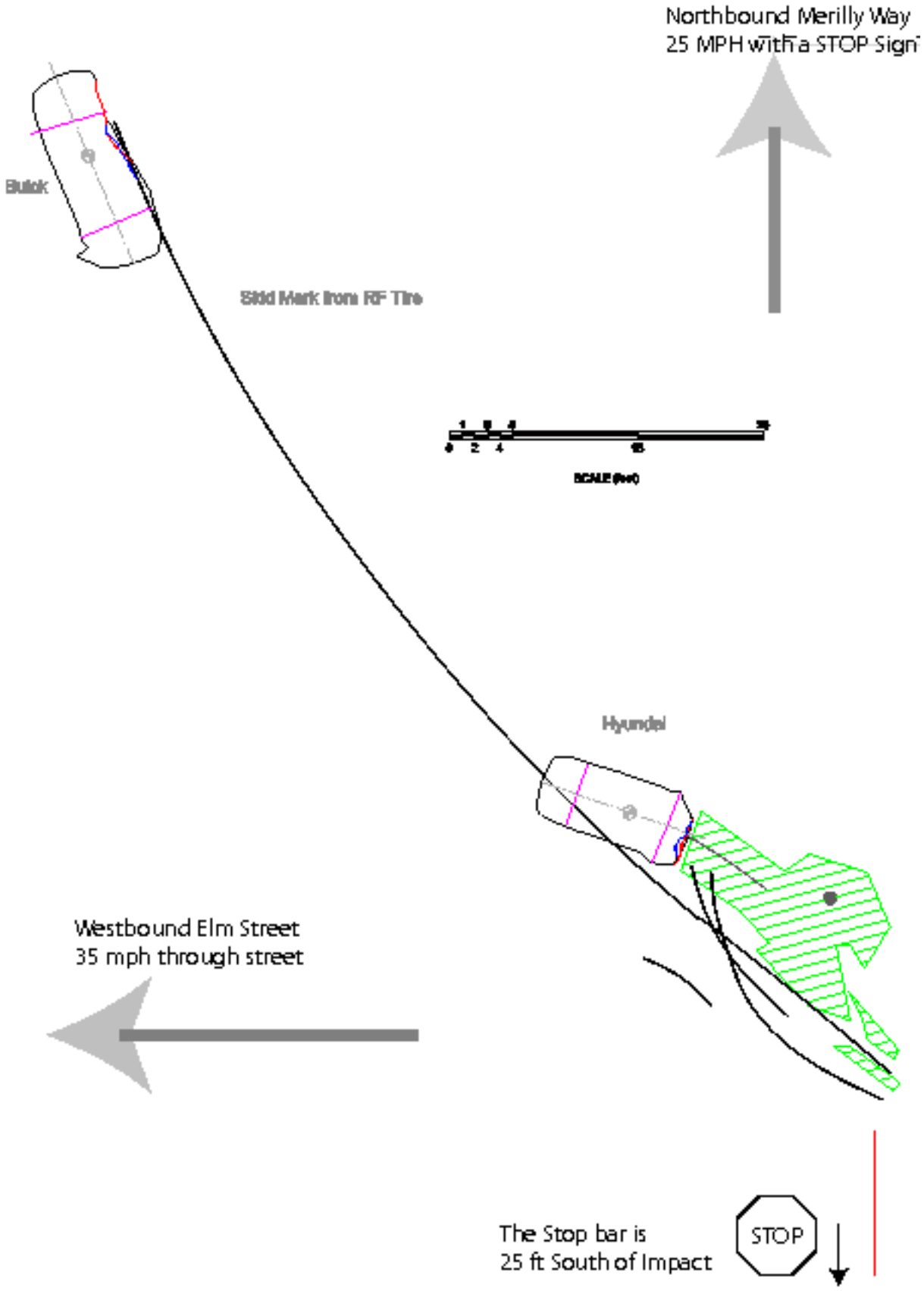


Figure 23.4: Scene drawing for crash scenario. Use this drawing to obtain distances and angles. Not shown are 37 ft of pre-impact scuff marks from the Hyundai indicating full braking.

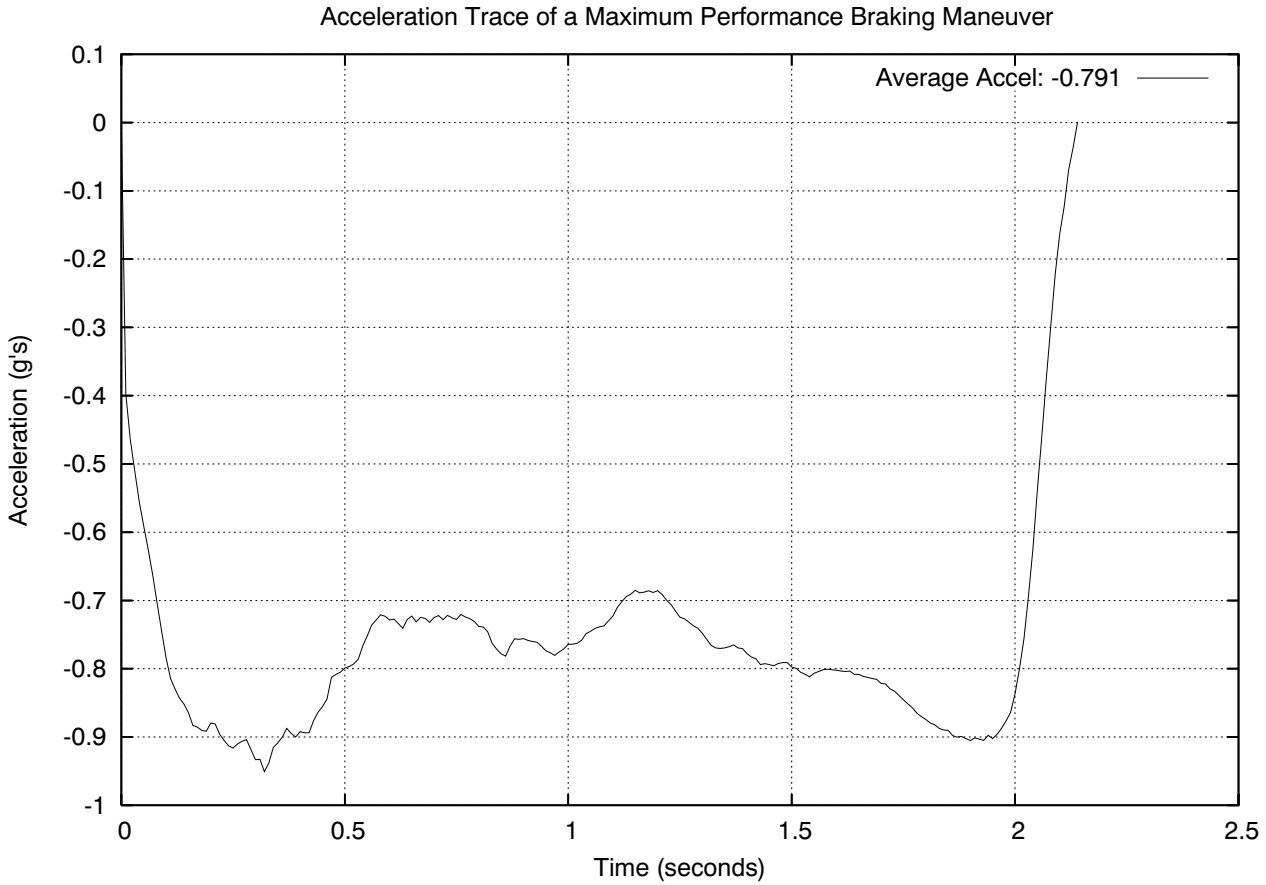


Figure 23.5: Representative accelerometer trace from skid test.

	Run 1	Run 2	Run 3	Run 4	Run 5
Average Acceleration (g's)	0.791	0.812	0.789	0.756	0.804

Table 23.1: Summary of acceleration values.

23.3.2 Friction Determination

The Buick and the Hyundai has similar composition rubber tires. An example vehicle with similar tire compounds was obtained and 5 braking tests were performed. A representative graph of the accelerometer trace are shown in Fig. 23.5 and summarized in Table 23.1. The road was level in both directions.

Based on the 5 tests, determine a statistical range of the likely value during the time of the crash. Keep in mind, the post impact path of the Buick showed only the right front wheel was braking. Therefore, the total friction force is reduced from a full locked wheel skid.

23.3.3 Vehicle Information

A handout will be provided in class based on the Expert AutoStats.

23.4 Assignment

Break into 2 teams comprising at least 1 engineering student and 1 law student each (4-6 people total). By mutual agreement or a flip of a coin, decide which team will represent the State and prosecute Ms. Smith for the manslaughter charge and which team will defend her. Each engineering student is to determine both vehicle speeds and determine if it was possible for Ms. Smith to have come to a complete stop at the intersection. Engineering students must work with the law students to prepare a 3-page written report summarizing their findings. After at least 2-weeks, the group will meet to exchange depositions. The prosecuting law student will depose the engineering student working for the defense for 30 min. Then, the law student working on the defense will depose the engineering student working for the prosecutor. The depositions will be video recorded and evaluated by faculty and researchers for consistency and accuracy.

23.5 Ethics for Accident Investigation and Reconstruction

A handout will be provided in class based on SAE J2314.

23.6 Reporting Requirements

A summary of the opinions regarding this case must be turned in. The following opinions should be given:

1. How did the vehicles come together. What were there orientations.
2. Speed of the Hyundai at impact.
3. Speed of the Buick at impact.
4. Speed of the Hyundai at the beginning of its skid.
5. Determination of the Buick at the beginning of the skid.
6. The change in velocity of the Buick.
7. The change in velocity of the Hyundai.
8. The Principal Direction of Force (PDOF) of the collision on the Buick (i.e. the direction of the impulse vector).
9. The Principal Direction of Force (PDOF) of the collision on the Buick (i.e. the direction of the impulse vector).
10. Who was at fault? Why did the crash occur?

24 Ansys Workbench Exercise

24.1 Objective

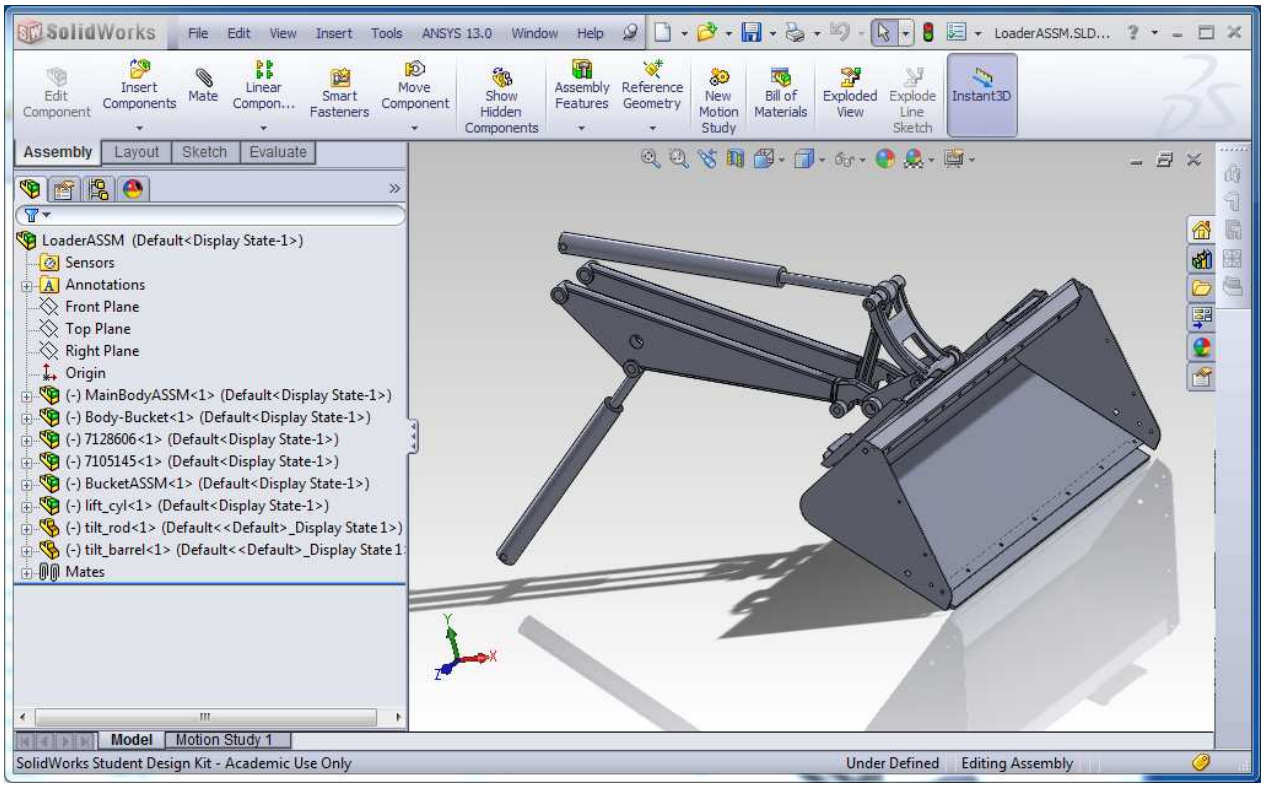
The goal of this lab is to introduce the student to modern engineering tools for analysis. These computer programs are quite powerful, but only if applied correctly. Therefore, it is important to understand what the computer is doing by working out a simple problem by hand first. Generally the following procedure should take place:

1. Develop an understanding of a problem by solving a special case by hand.
2. Solve the same problem using the engineering software. Ensure the numbers match the hand calculations.
3. Modify the computer model to solve more complex problems. Understand the limitations of adding the complexities.

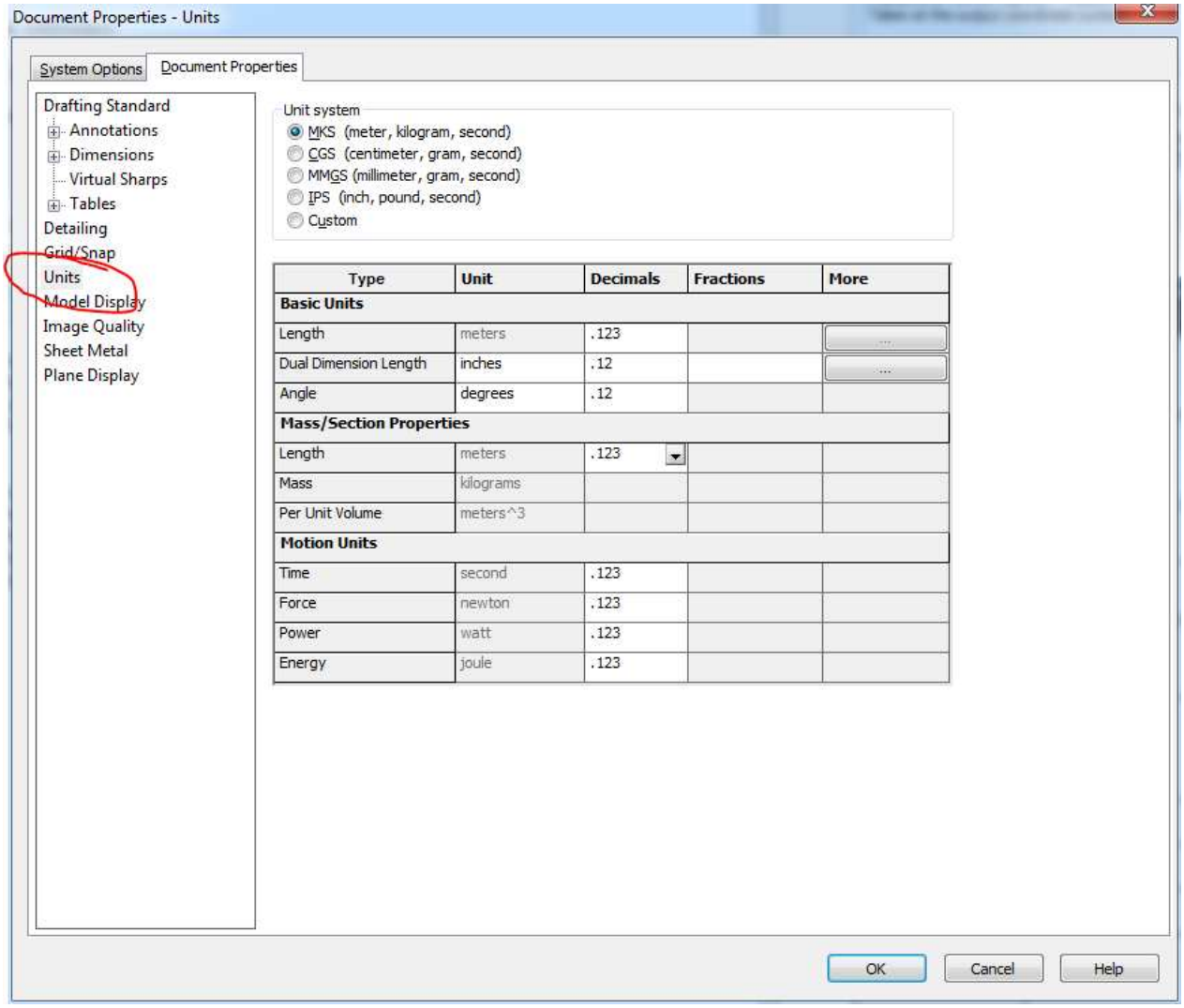
24.2 Theory and Hand Calculations

You have been asked to determine the dynamic forces in a Bobcat front loader. A complete Solid-Works (version 2011-2012) has been created of the front loader assembly. You must detemine the internal joint forces of the mechanism while it is in motion. Assume the lifting cylinder is expanding at a constant rate of 5 cm/s. The tilting cylinder (top) is fixed.

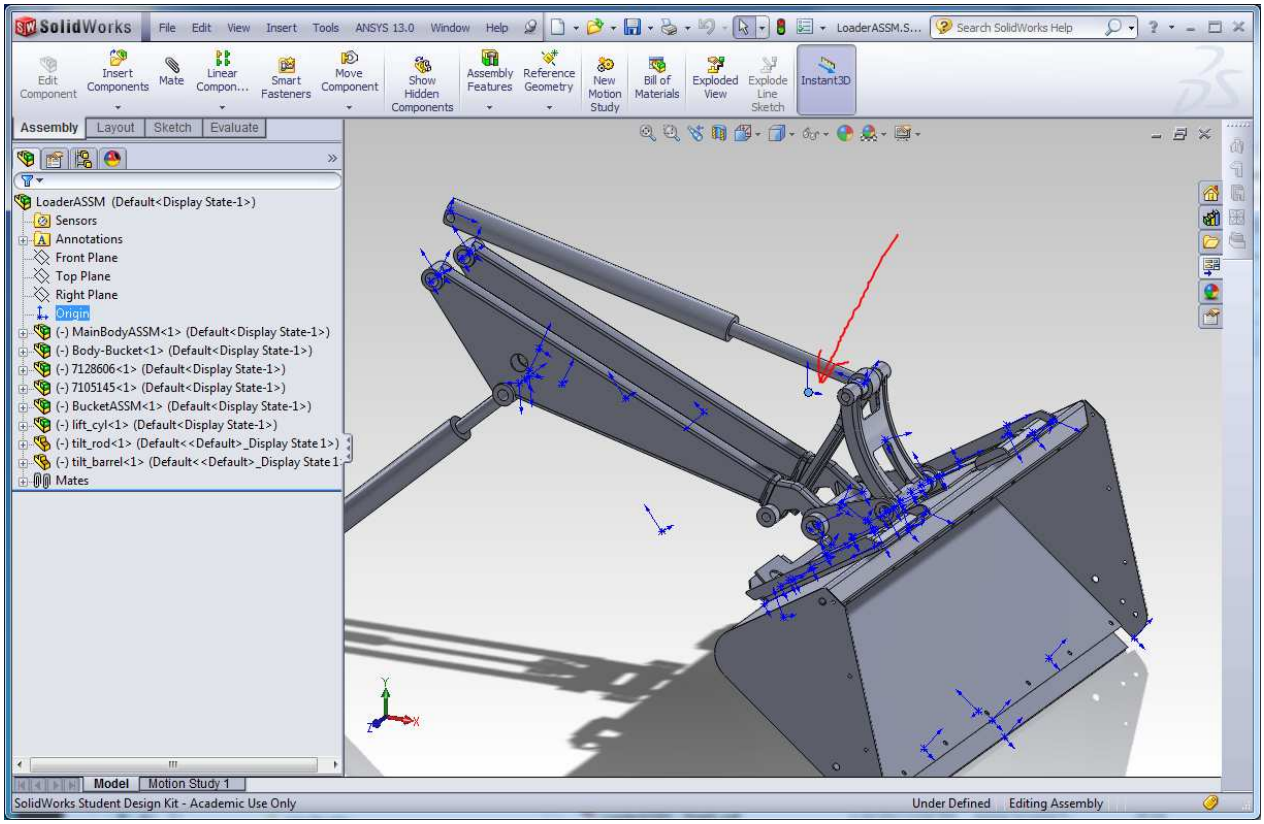
1. Navigate to the Shared drive and find the following files S:\ENS\Mechanical Engineering\Daily\Machine Dynamics Labs\Ansys Project for Bobcat\
2. Copy files to a local directory or your W: drive.
3. Open the LoaderASSM.SLDASM file in SolidWorks 2011-2012 or newer.



4. Select the MKS system of units to work with by entering the Tools -> Options. Make the Length parameter in the Mass/Section Properties 6 digits.

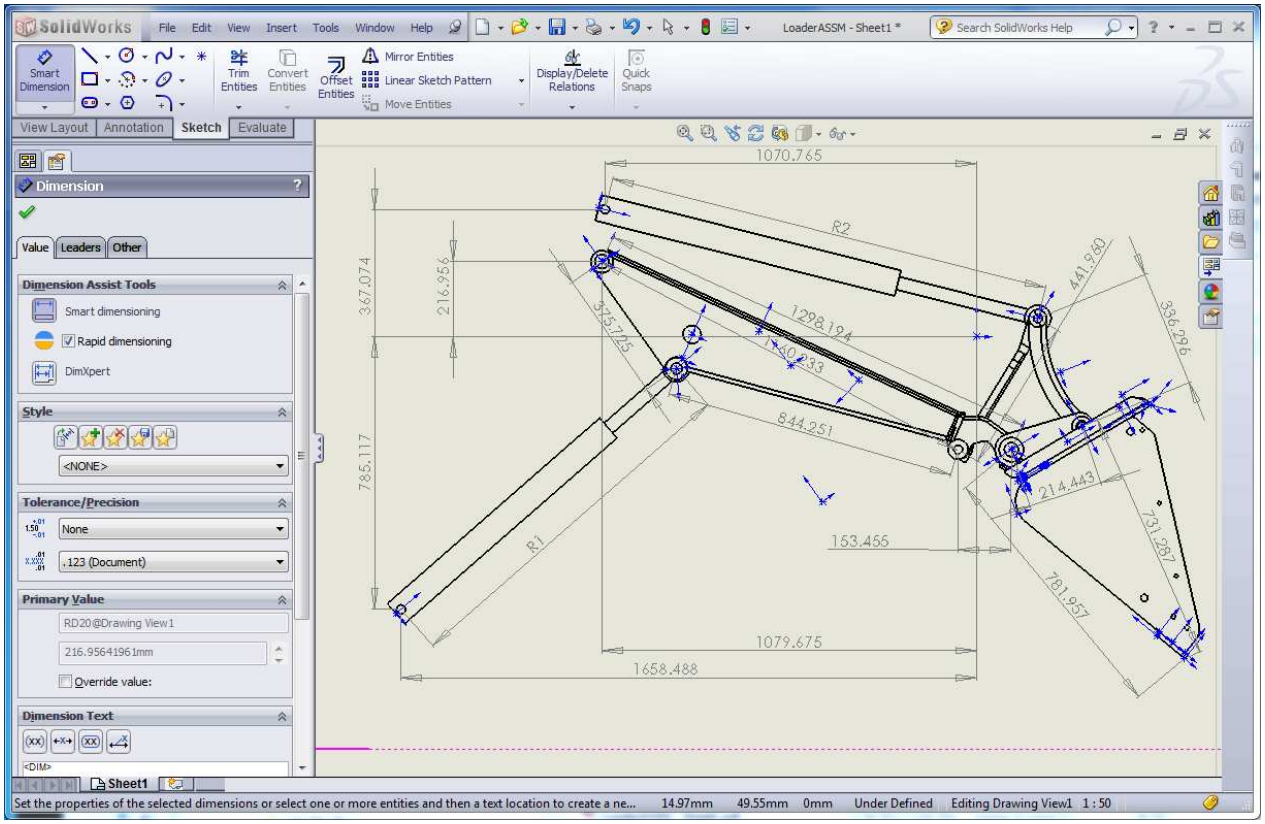


5. Locate the origin of the model. origins can be turned on under the View Menu.



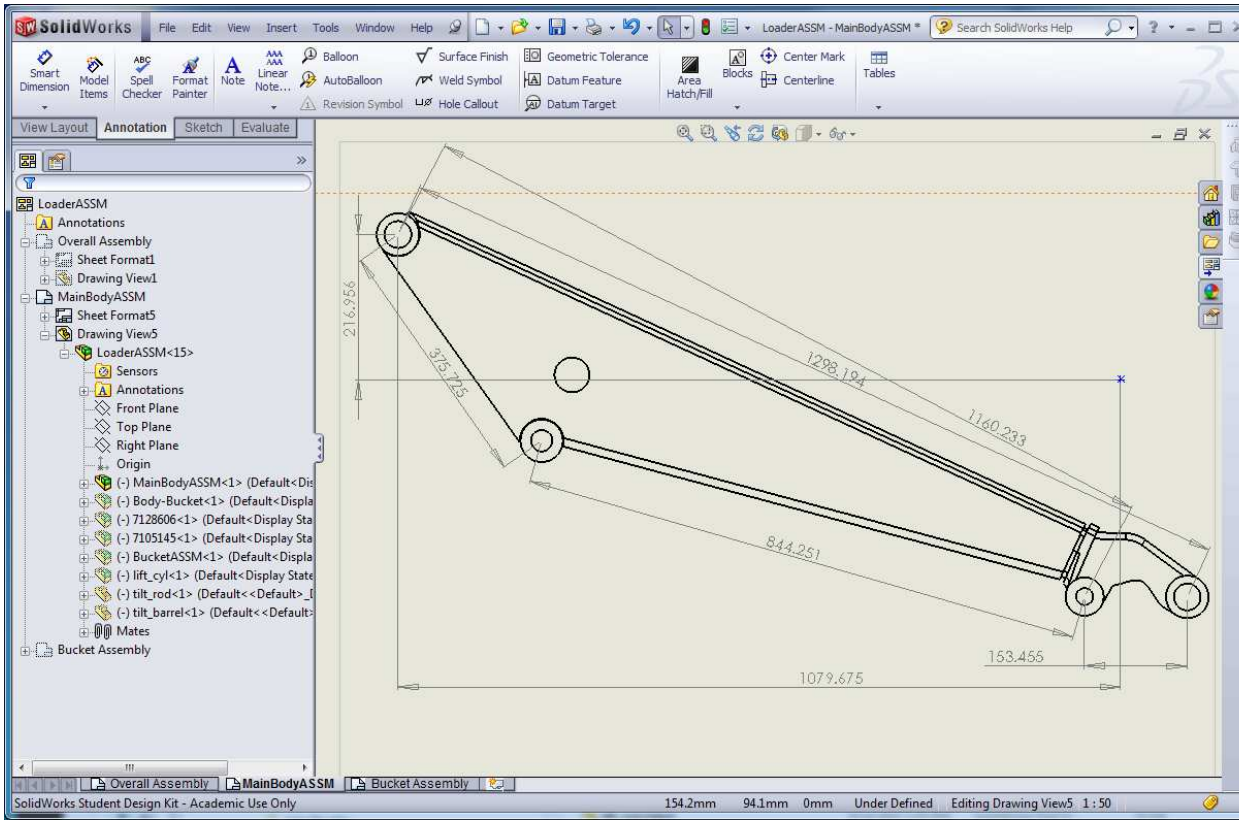
6. Create a drawing from the model by selecting File -> Make Drawing from Assembly. Use a 1:10 scale.

7. Add the appropriate dimensions to the drawing. All dimensions should be aligned except those emenating from the orgin. Print the assembly drawing.

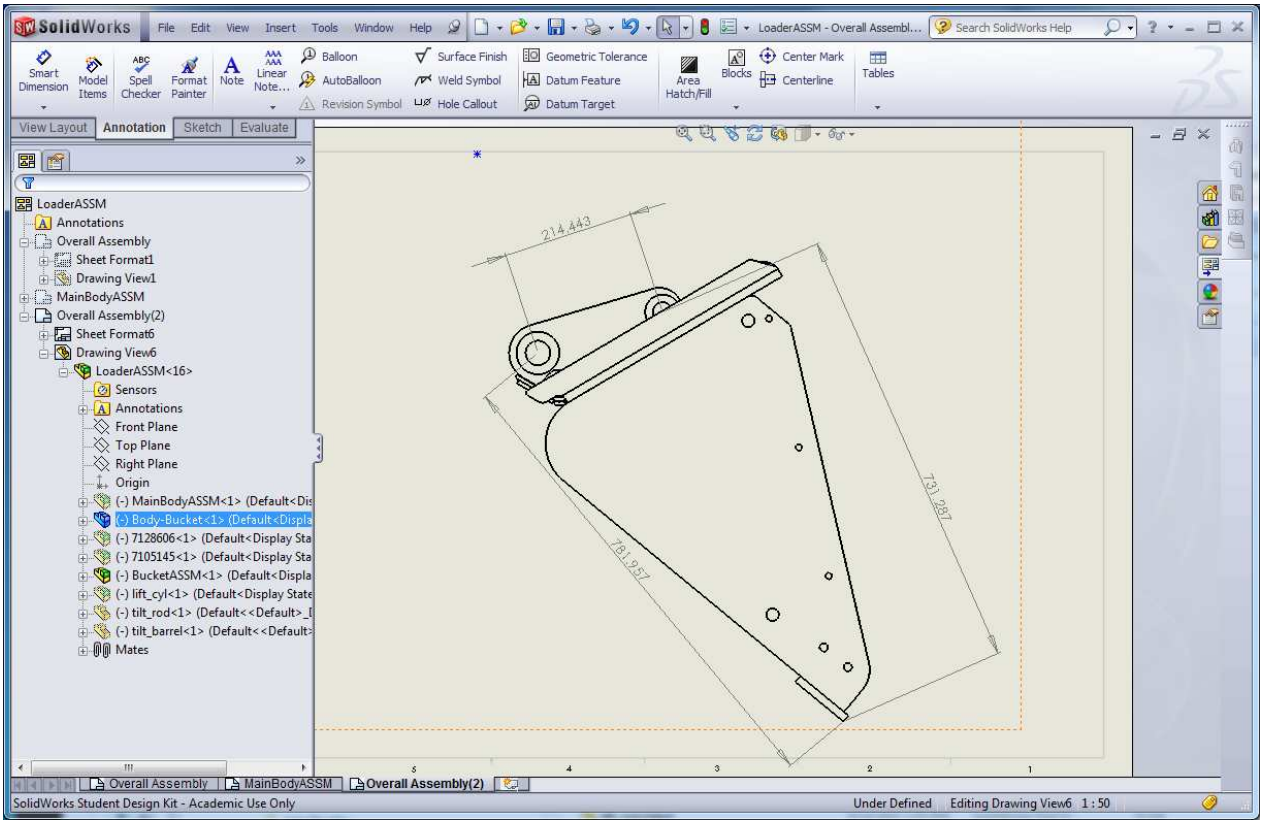


8. Add another sheet to draw individual parts. You can copy and paste Sheet 1. Turn off all components except what is of interest. Also, turn off non relevant dimensions. Print these pages to help develop the equations of motion.

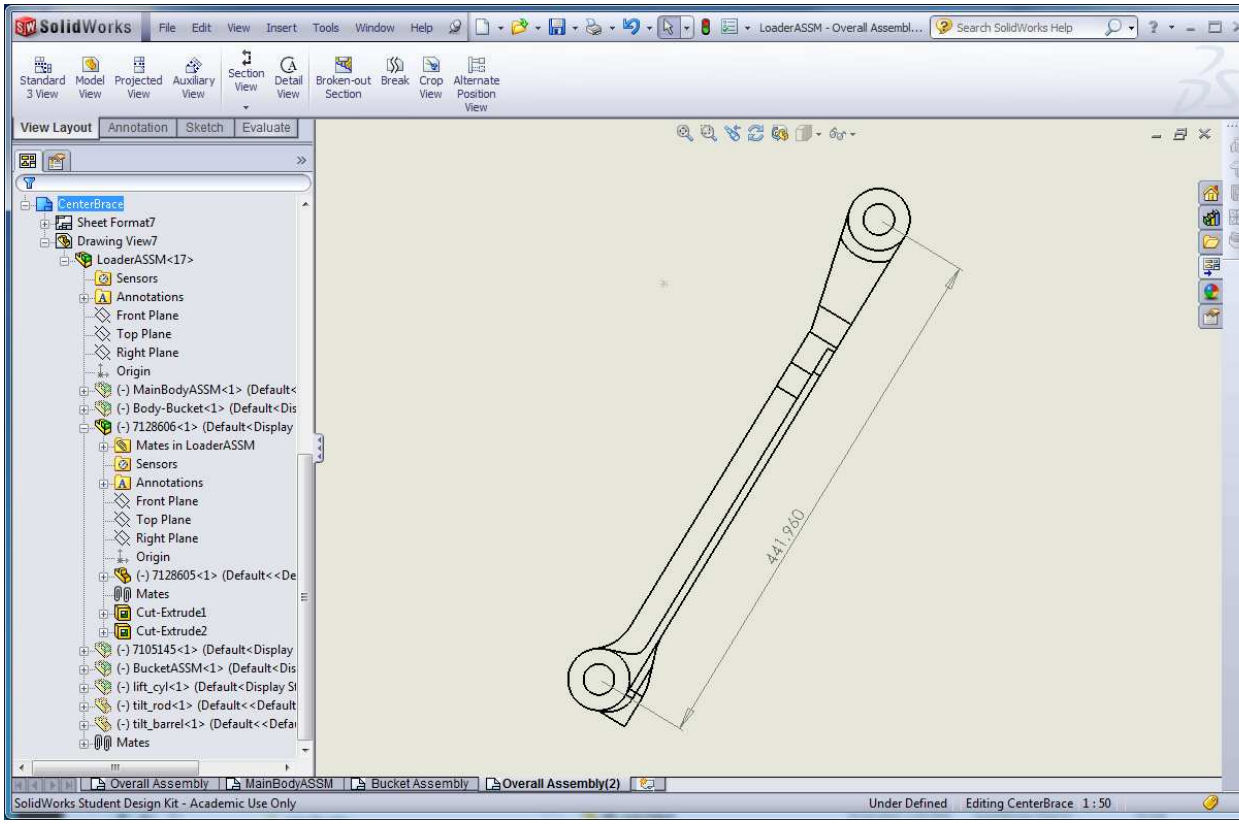
a) Print the MainBodyASSM drawing.



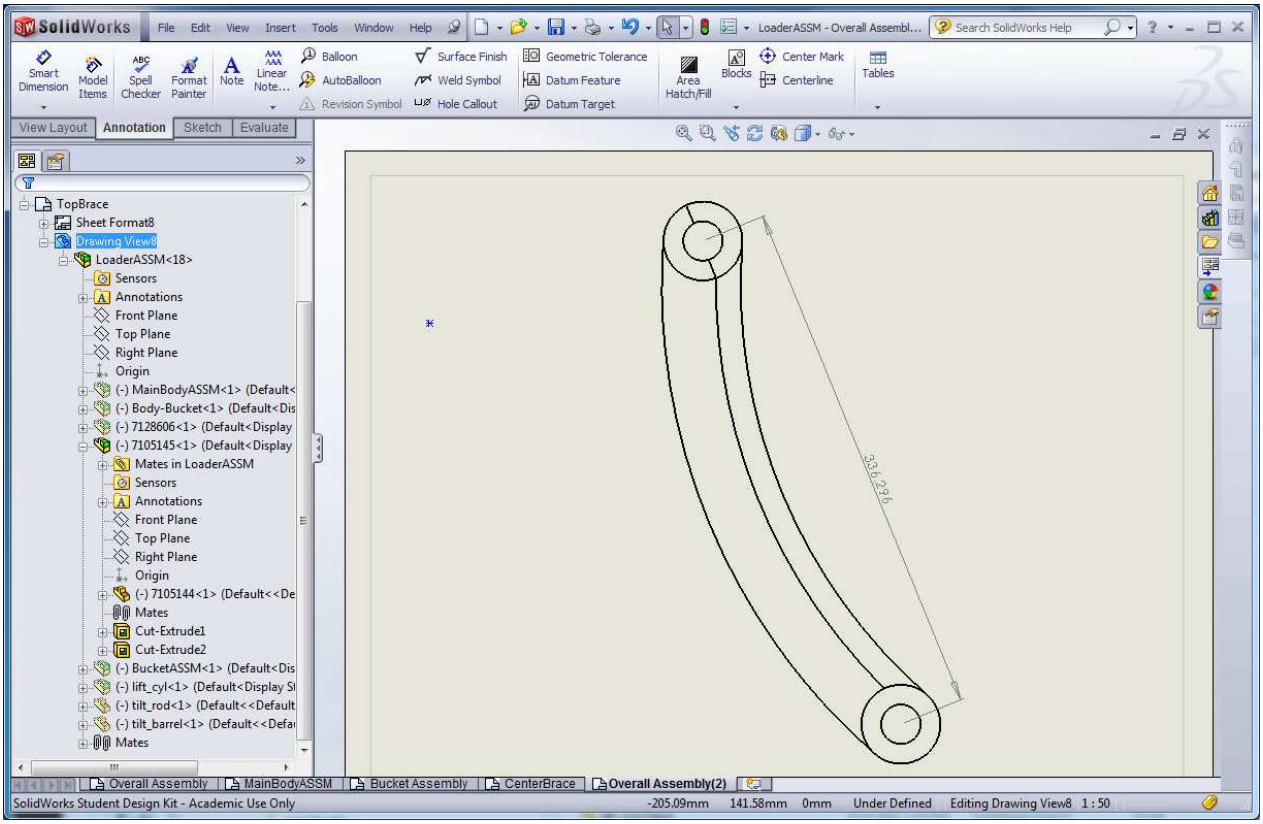
b) Print the Body-Bucket and BucketASSM drawing.



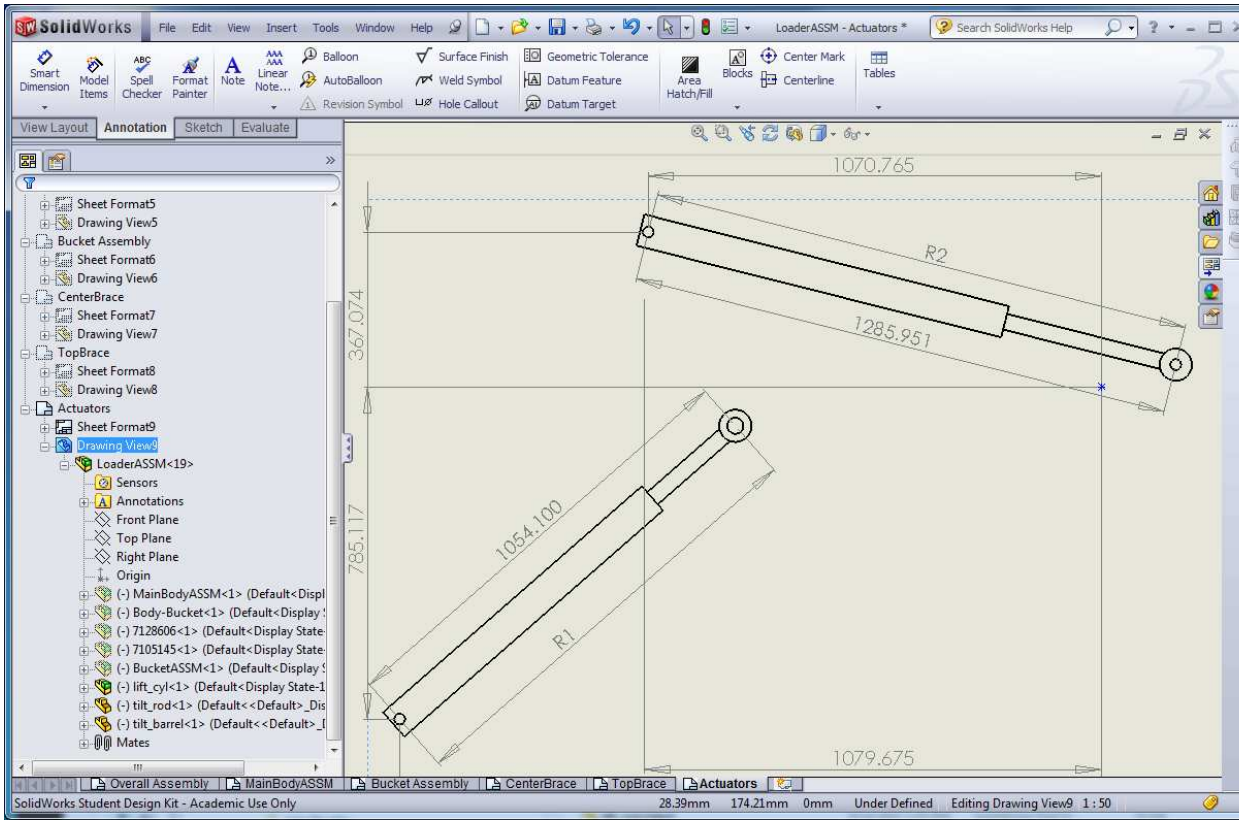
c) Print the Center brace drawing.



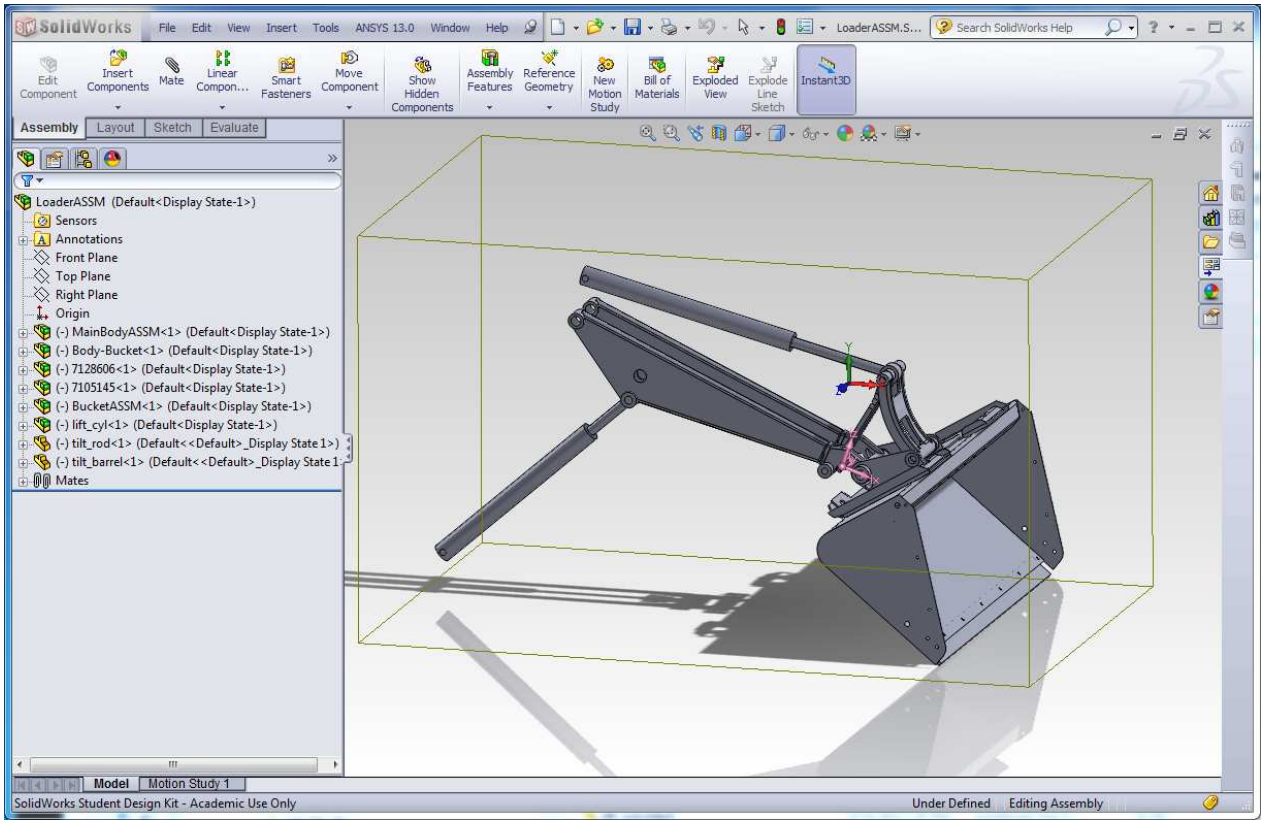
d) Print the Top brace drawing.



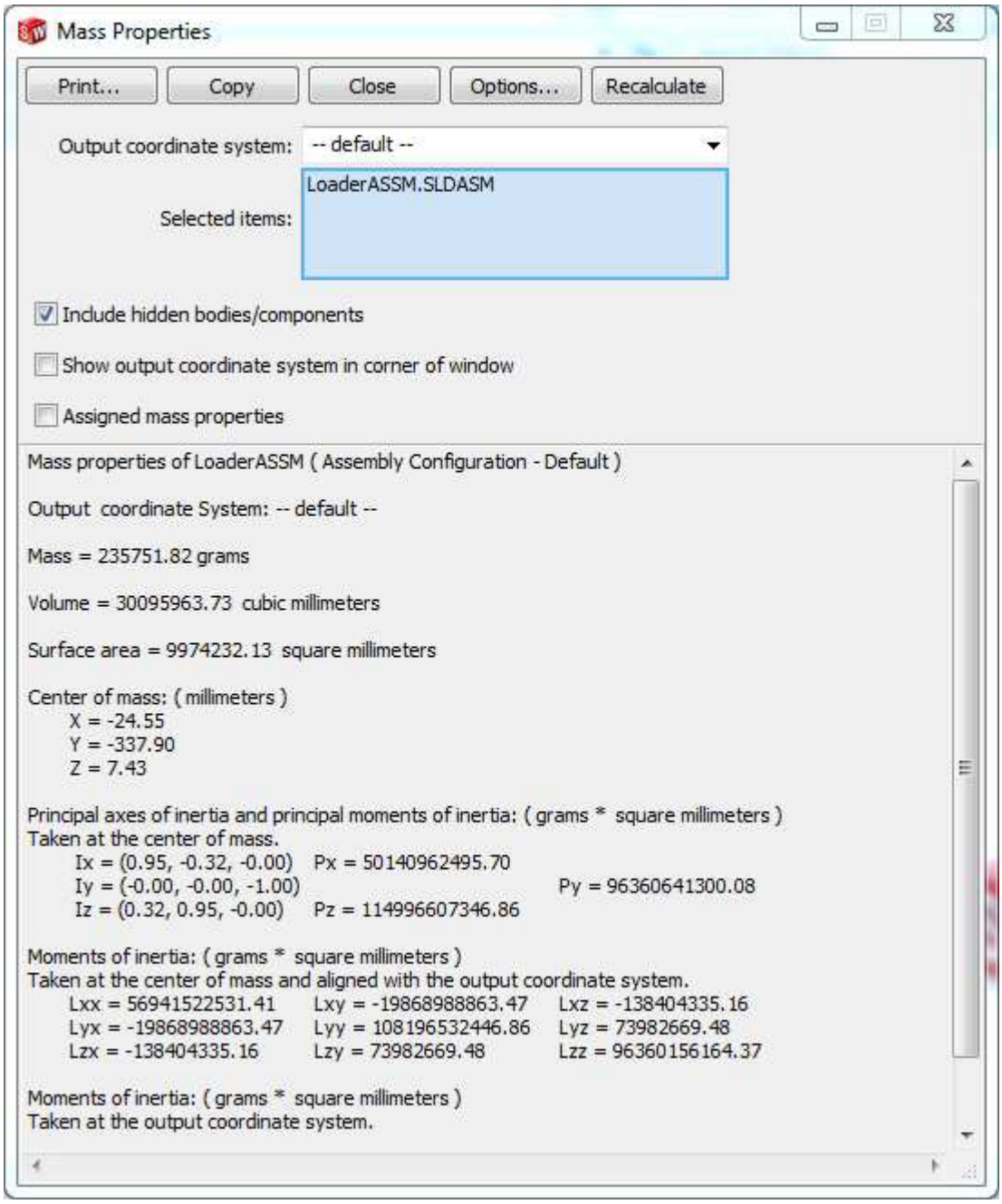
e) Print a drawing of the Actuators.



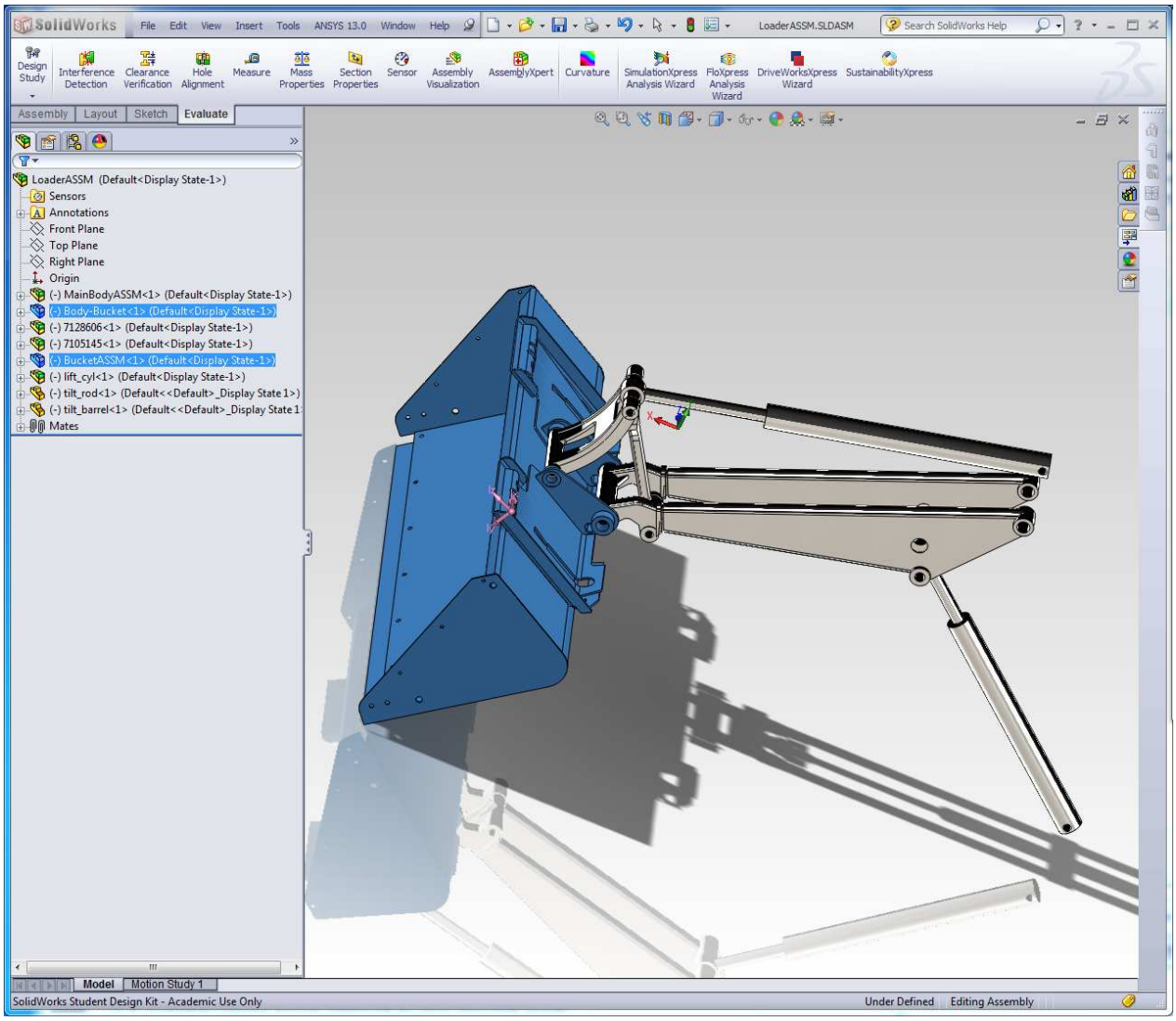
9. From the model assembly, open Tools -> Mass Properties. Note the origin and the Center of Gravity (CG) arrows.

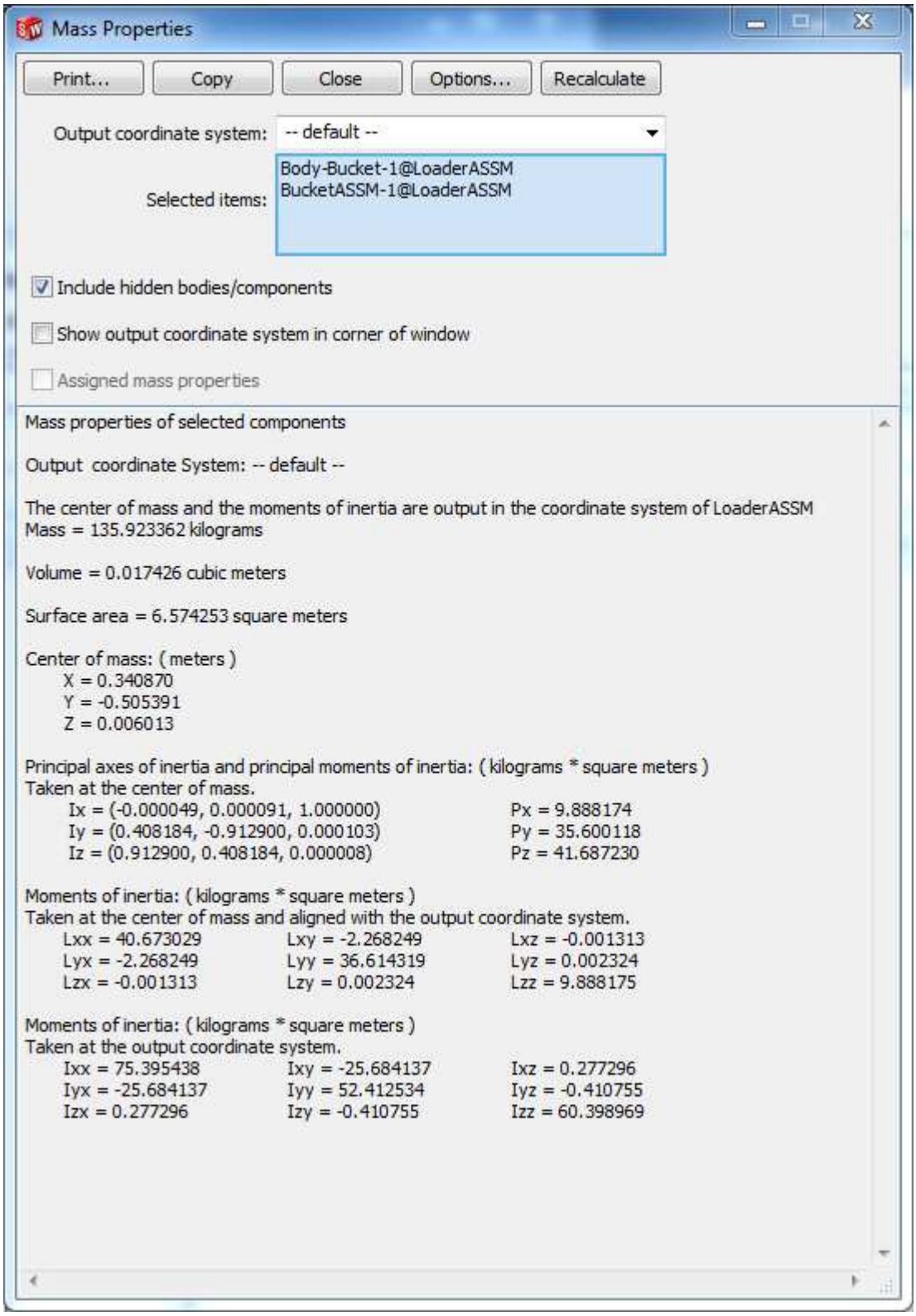


10. A window should appear that enables you to select different parts. Use this window to querey the mass properties of all the parts. The loader is made from steel. Verify the density of steel is correct in the model.


Mass Properties
Print...
Copy
Close
Options...
Recalculate
Output coordinate system:
-- default --
Selected items:
LoaderASSM.SLDASM
Include hidden bodies/components
Show output coordinate system in corner of window
Assigned mass properties
Mass properties of LoaderASSM (Assembly Configuration - Default)
Output coordinate System: -- default --
Mass = 235751.82 grams
Volume = 30095963.73 cubic millimeters
Surface area = 9974232.13 square millimeters
Center of mass: (millimeters)
X = -24.55
Y = -337.90
Z = 7.43
Principal axes of inertia and principal moments of inertia: (grams * square millimeters)
Taken at the center of mass.
Ix = (0.95, -0.32, -0.00) Px = 50140962495.70
Iy = (-0.00, -0.00, -1.00) Py = 96360641300.08
Iz = (0.32, 0.95, -0.00) Pz = 114996607346.86
Moments of inertia: (grams * square millimeters)
Taken at the center of mass and aligned with the output coordinate system.
Lxx = 56941522531.41 Lxy = -19868988863.47 Lxz = -138404335.16
Lyx = -19868988863.47 Lyy = 108196532446.86 Lyz = 73982669.48
Lzx = -138404335.16 Lzy = 73982669.48 Lzz = 96360156164.37
Moments of inertia: (grams * square millimeters)
Taken at the output coordinate system.

11. For each linkage, determine the mass, center of mass location, and moment of inertia about the CG by ensuring that the Mass Properties windo has only the components of interest highlighted and press the Recalculate button. For example:


SolidWorks
File Edit View Insert Tools ANSYS 13.0 Window Help
LoaderASSM.SLDASM
Search SolidWorks Help
Design Study
Interference Detection
Clearance Verification
Hole Alignment
Measure
Mass Properties
Section Properties
Sensor
Assembly Visualization
AssemblyXpert
Curvature
SimulationXpress Analysis Wizard
FloXpress Analysis Wizard
DriveWorksXpress Wizard
SustainabilityXpress
Assembly
Layout
Sketch
Evaluate
LoaderASSM (Default<Display State-1>)
Sensors
Annotations
Front Plane
Top Plane
Right Plane
Origin
(-) MainBodyASSM<1> (Default<Display State-1>)
(-) Body-Bucket<1> (Default<Display State-1>)
(-) 7128606<1> (Default<Display State-1>)
(-) 7105145<1> (Default<Display State-1>)
(-) BucketASSM<1> (Default<Display State-1>)
(-) lift_cyl<1> (Default<Display State-1>)
(-) tilt_rod<1> (Default<<Default>_Display State 1>)
(-) tilt_barrel<1> (Default<<Default>_Display State 1
Mates
Model
Motion Study 1
SolidWorks Student Design Kit - Academic Use Only
Under Defined
Editing Assembly


Mass Properties
Print...
Copy
Close
Options...
Recalculate
Output coordinate system: -- default --
Selected items:
Body-Bucket-1@LoaderASSM
BucketASSM-1@LoaderASSM
Include hidden bodies/components
Show output coordinate system in corner of window
Assigned mass properties
Mass properties of selected components
Output coordinate System: -- default --
The center of mass and the moments of inertia are output in the coordinate system of LoaderASSM
Mass = 135.923362 kilograms
Volume = 0.017426 cubic meters
Surface area = 6.574253 square meters
Center of mass: (meters)
X = 0.340870
Y = -0.505391
Z = 0.006013
Principal axes of inertia and principal moments of inertia: (kilograms * square meters)
Taken at the center of mass.
Ix = (-0.000049, 0.000091, 1.000000) Px = 9.888174
Iy = (0.408184, -0.912900, 0.000103) Py = 35.600118
Iz = (0.912900, 0.408184, 0.000008) Pz = 41.687230
Moments of inertia: (kilograms * square meters)
Taken at the center of mass and aligned with the output coordinate system.
Lxx = 40.673029 Lxy = -2.268249 Lxz = -0.001313
Lyx = -2.268249 Lyy = 36.614319 Lyz = 0.002324
Lzx = -0.001313 Lzy = 0.002324 Lzz = 9.888175
Moments of inertia: (kilograms * square meters)
Taken at the output coordinate system.
Ixx = 75.395438 Ixy = -25.684137 Ixz = 0.277296
Iyx = -25.684137 Iyy = 52.412534 Iyz = -0.410755
Izx = 0.277296 Izy = -0.410755 Izz = 60.398969

12. Save your file.

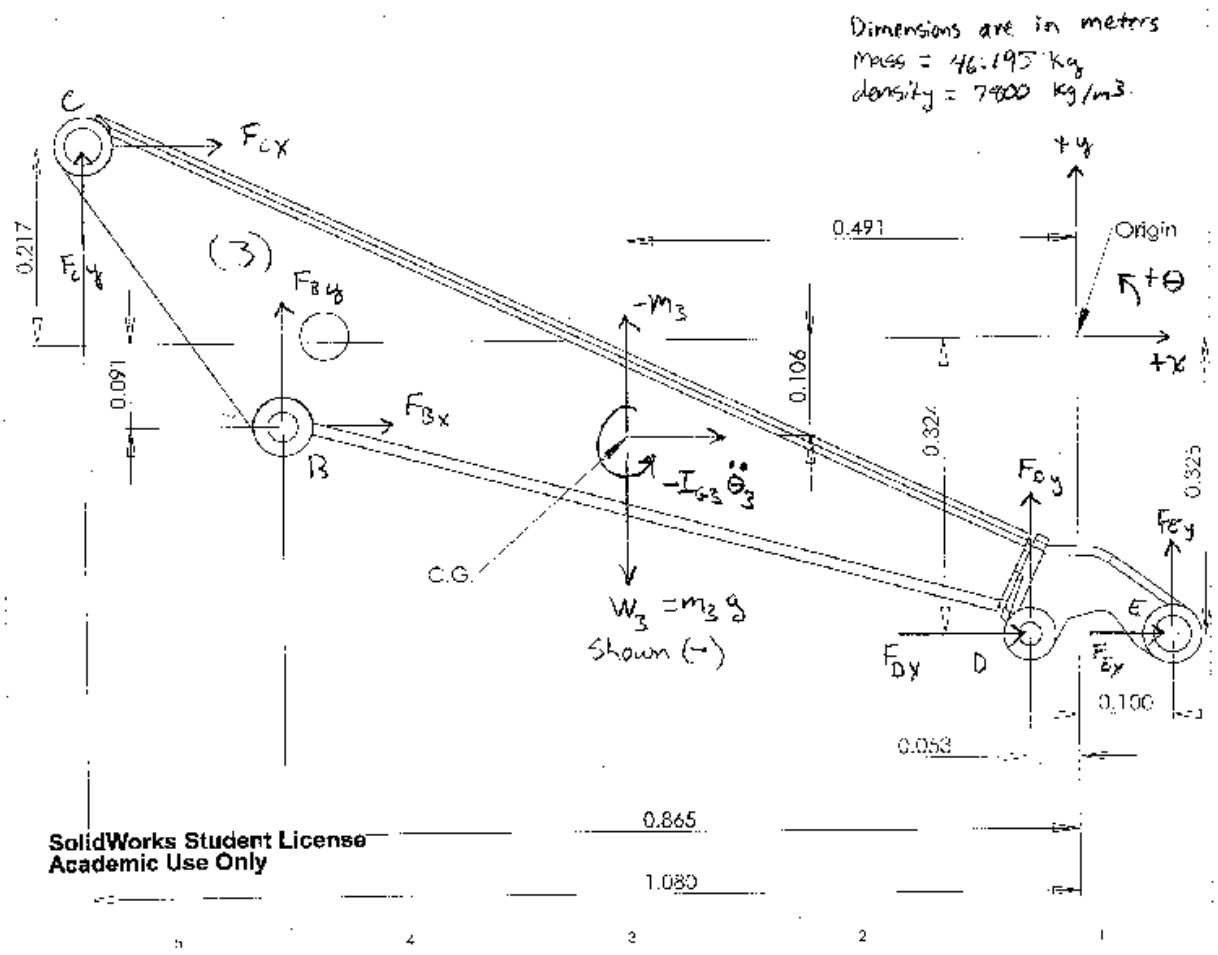
13. Fill in the table below:

Component	Mass	CG-X	CG-Y	Dist from CG	$I_{CG,zz}$	$I_O = I_{CG} + mr^2$
Main Body ASSM						
BucketASSM and Body-Bucket						
Center Brace						
Top Brace						
Lift Cylinder						
Tilt rod and barrel						

14. Label all the joints with letters on all drawings. You should have A-H.

15. Label all links with numbers. You should have 6 links.

16. Turn the prints of the individual components into Free Body Diagrams by adding forces at the pin joints, the Center of mass with a weight load, and all the D'Alembert forces. For example, a free body diagram for the main link is as follows.



17. Using the free body diagrams, sum the forces and moments to create 3 equations for each link. This will generate 18 equations. For example, the three equations for the main link are as follows:

$$\sum F_x: \quad F_{Cx} + F_{Bx} + F_{Dx} + F_{Ex} - m_3 a_{3x} = 0$$

$$\sum F_y: \quad F_{Cy} + F_{By} + F_{Dy} + F_{Ey} - m_3 a_{3y} - w_3 = 0$$

$$\sum M_o: \quad -0.217F_{Cx} + 0.091F_{Bx} + 0.324F_{Dx} + 0.325F_{Ex} - 0.106m_3 a_{3x}$$
$$-1.080F_{Cy} - 0.865F_{By} - 0.053F_{Dy} + 0.100F_{Ey} + 0.491m_3 a_{3y} + 0.491w_3 - I_{G3}\ddot{\theta}_3 = 0$$

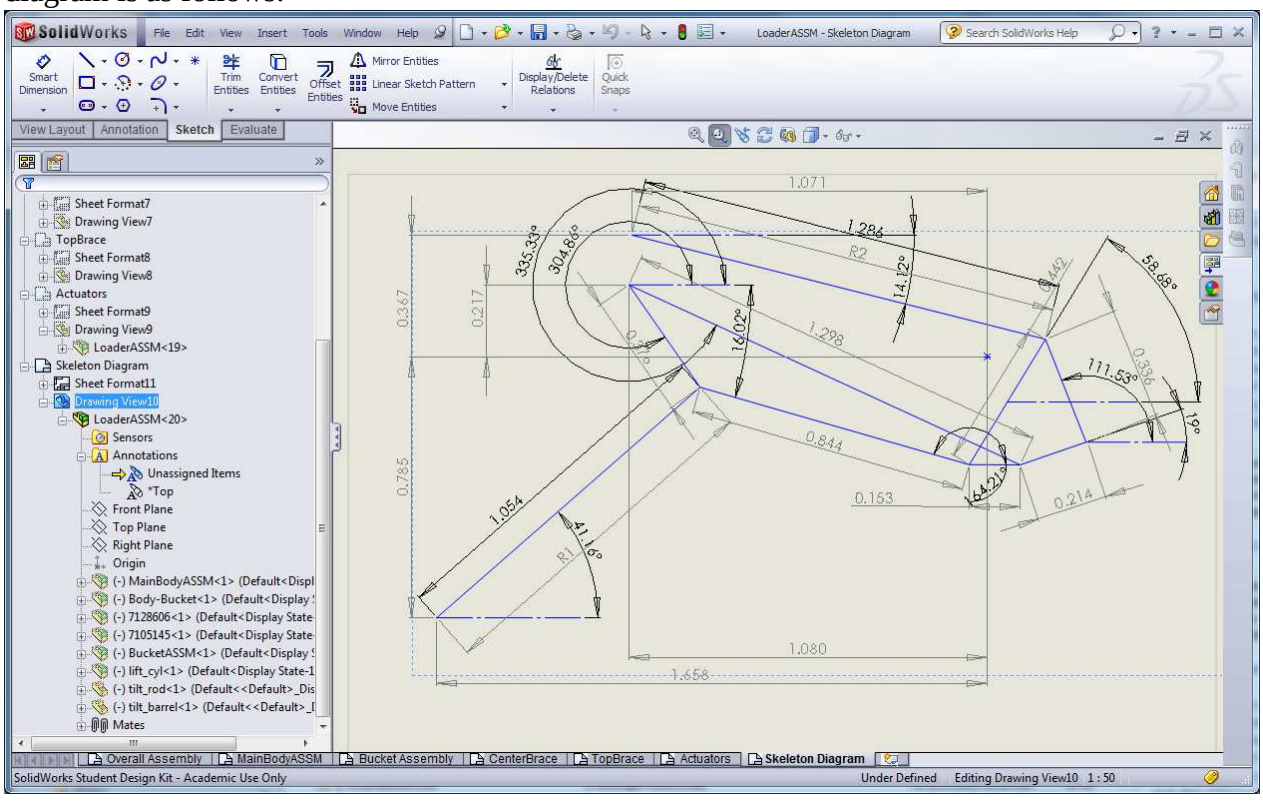
18. Notice that there is one junction with three links. The sum of the forces for this joint need to

be equal. If that joint is labeled G and connects links 2, 4 and 5, then

$$F_{Gx}^{(2)} + F_{Gx}^{(4)} + F_{Gx}^{(5)} = 0$$

$$F_{Gy}^{(2)} + F_{Gy}^{(4)} + F_{Gy}^{(5)} = 0$$

19. Assume the bottom lifting link is expanding at a constant rate of 0.05 meters/second at the instant shown. Assume the top tilting link is fixed. From mechanisms class you can construct the loop closure equations and obtain relationsips for velocity and accelration. A skeleton diagram is as follows:



A pdf of all these sketches and dimensions are availaible in the zipped file on the S: drive.

20. A kinematic analysis results in the following velocities (rad/s):

$$\dot{\theta}_1 = -0.00516532$$

$$\dot{\theta}_2 = 0.126118$$

$$\dot{\theta}_3 = 0.133765$$

$$\dot{\theta}_4 = 0.0869611$$

$$\dot{\theta}_5 = 0.0936874$$

$$\dot{\theta}_6 = 0.0559266$$

21. A kinematic analysis results in the following accelerations (rad/s/s):

$$\ddot{\theta}_1 = 0.00691378$$

$$\ddot{\theta}_2 = -0.000742446$$

$$\ddot{\theta}_3 = -0.00202351$$

$$\ddot{\theta}_4 = -0.00298996$$

$$\ddot{\theta}_5 = -0.00441184$$

$$\ddot{\theta}_6 = -0.00422151$$

    A Mathematica print available on the S: drive contains more kinematic values.

22. Using these kinematic values, set-up and solve the overall system of equations for the unknown forces and force constraints. There should be 20 equations and 20 unknowns with 18 equations from the free body diagrams of the links and 2 equations from the force constraint of Point G.

23. Determine the total forces (magnitude) on the joints. Fill out the table below:

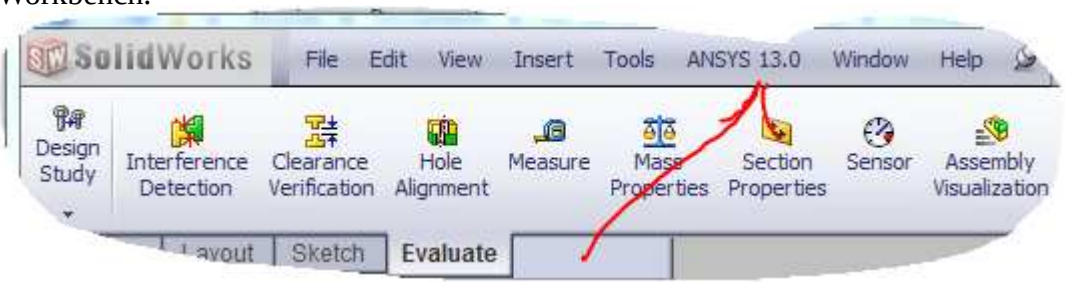
Joint	A	B	C	D	E	F	$G^{(2)}$	$G^{(4)}$	$G^{(5)}$	H
Force										

24.3 Ansys Exercise

This section describes the workflow to determine dynamic forces in Ansys Workbench 13.0.

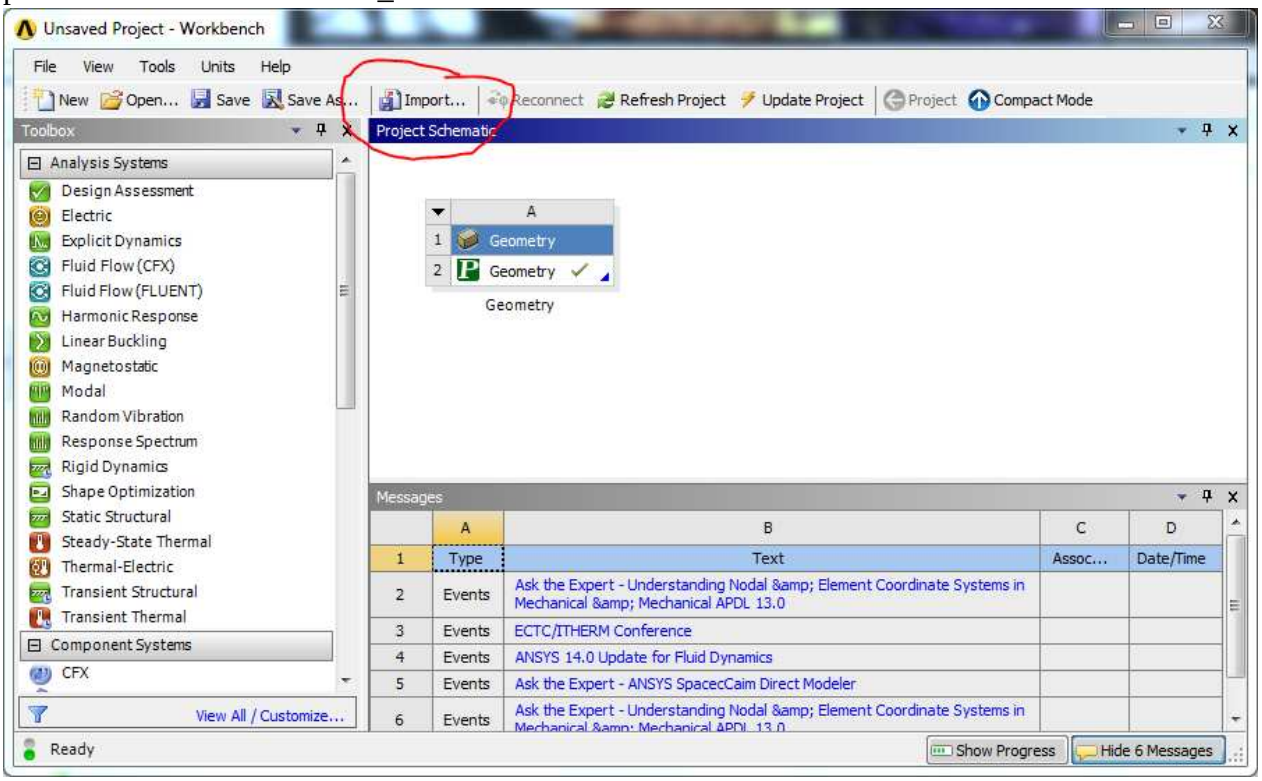
1. To Import Geometry, either

a) With the SolidWorks Assembly open, click on the Ansys 13.0 menu item and select Workbench.

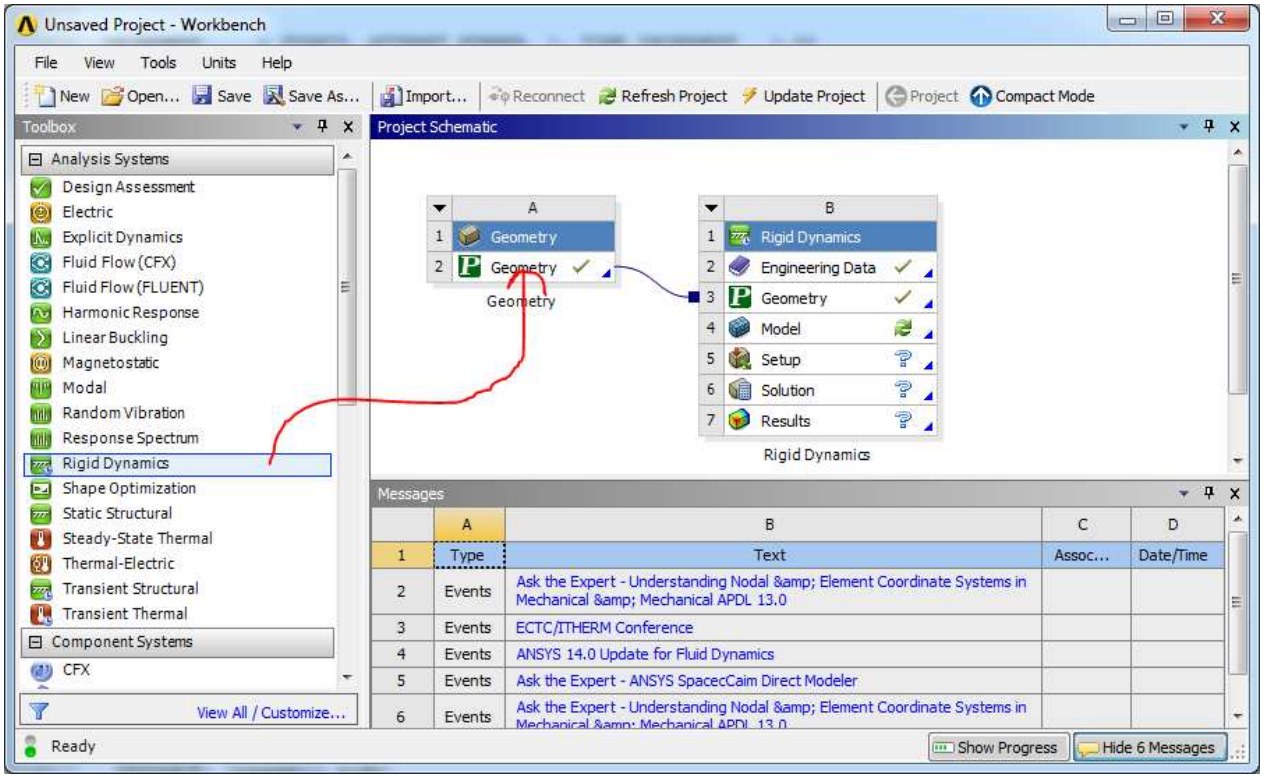


or

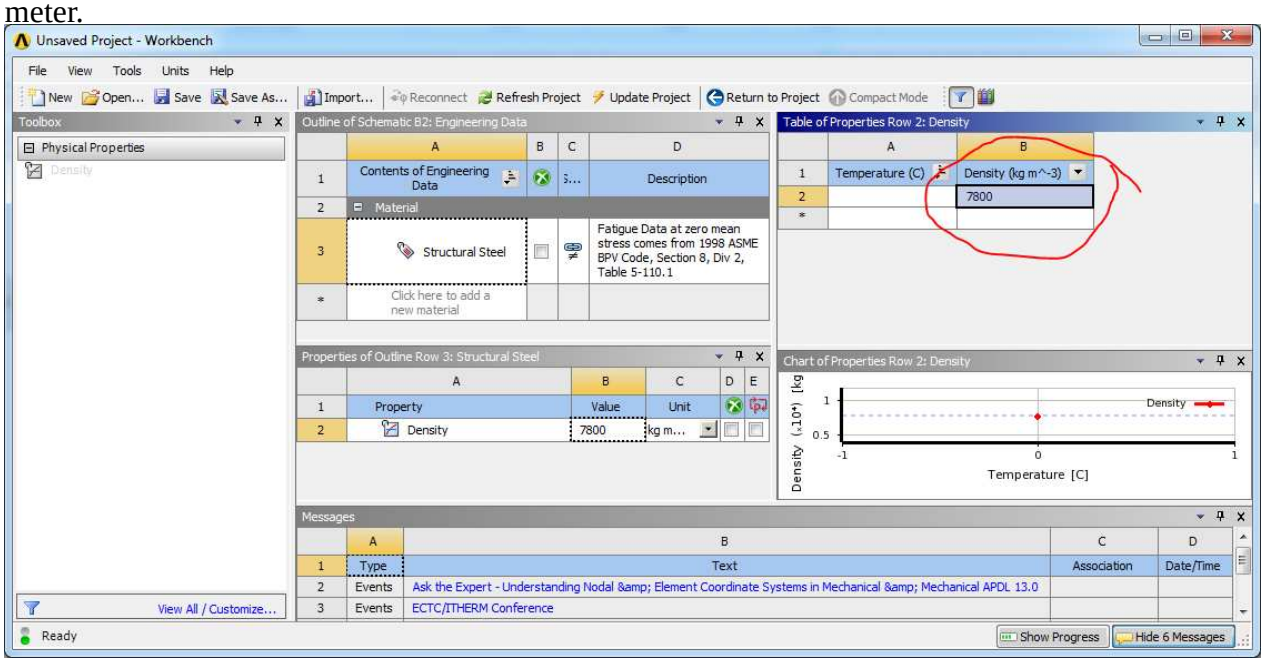
b) Open Ansys Workbench from the Windows Start menu and import geometry. Select the parasolid file LoaderASSM.x_t



2. Drag the Rigid Dynamics analysis system onto the Geometry cell.



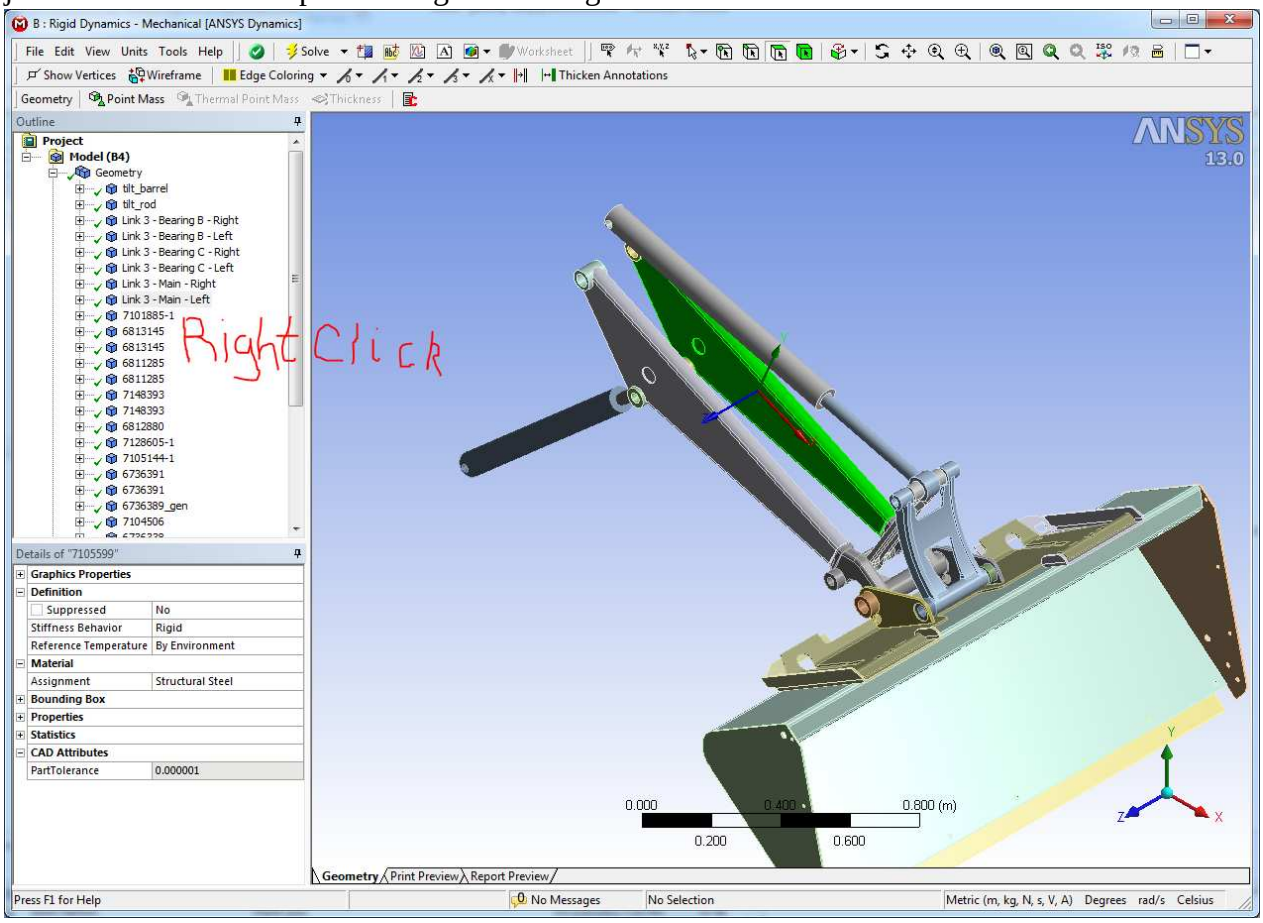
3. Double click on the Engineering Data and change the density of steel to 7800 kg per cubic meter.


Unsaved Project - Workbench
File View Tools Units Help
New Open... Save Save As... Import... Reconnect Refresh Project Update Project Return to Project Compact Mode
Toolbox
Physical Properties
Density
View All / Customize...
Outline of Schematic B2: Engineering Data
A B C D
1 Contents of Engineering Data Description
2 Material
3 Structural Steel Fatigue Data at zero mean stress comes from 1998 ASME BPV Code, Section 8, Div 2, Table 5-110.1
* Click here to add a new material
Table of Properties Row 2: Density
A B
1 Temperature (C) Density (kg m^-3)
2 7800
*
Properties of Outline Row 3: Structural Steel
A B C D E
1 Property Value Unit
2 Density 7800 kg m...
Chart of Properties Row 2: Density
Density (x10^) [kg
Density
Temperature [C]
Messages
A B C D
1 Type Text Association Date/Time
2 Events Ask the Expert - Understanding Nodal & Element Coordinate Systems in Mechanical & Mechanical APDL 13.0
3 Events ECTC/ITHERM Conference
Ready
Show Progress
Hide 6 Messages

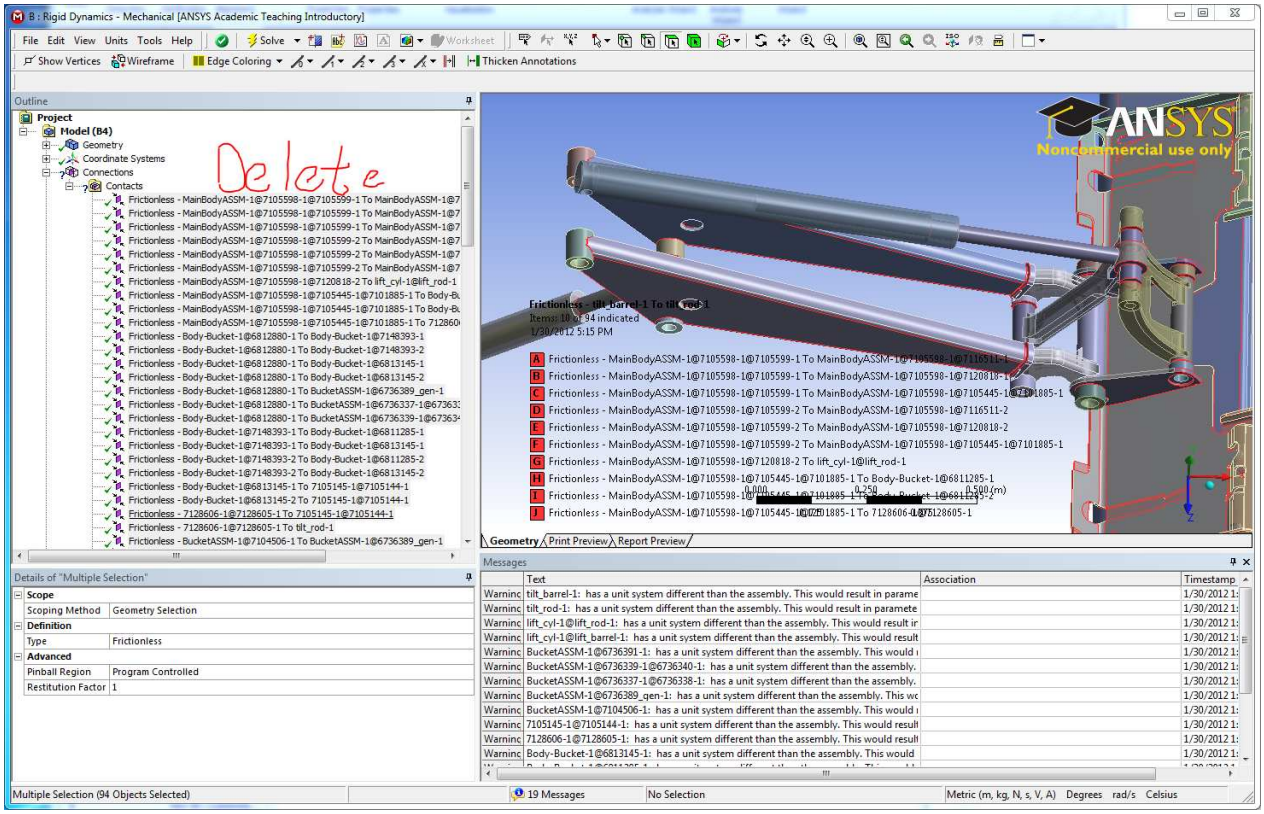
Return to the Project when finished.

4. Double click the Model cell under the Rigid Dynamics. This will open Ansys Mechanical and process the solid geometry. This may take a while.

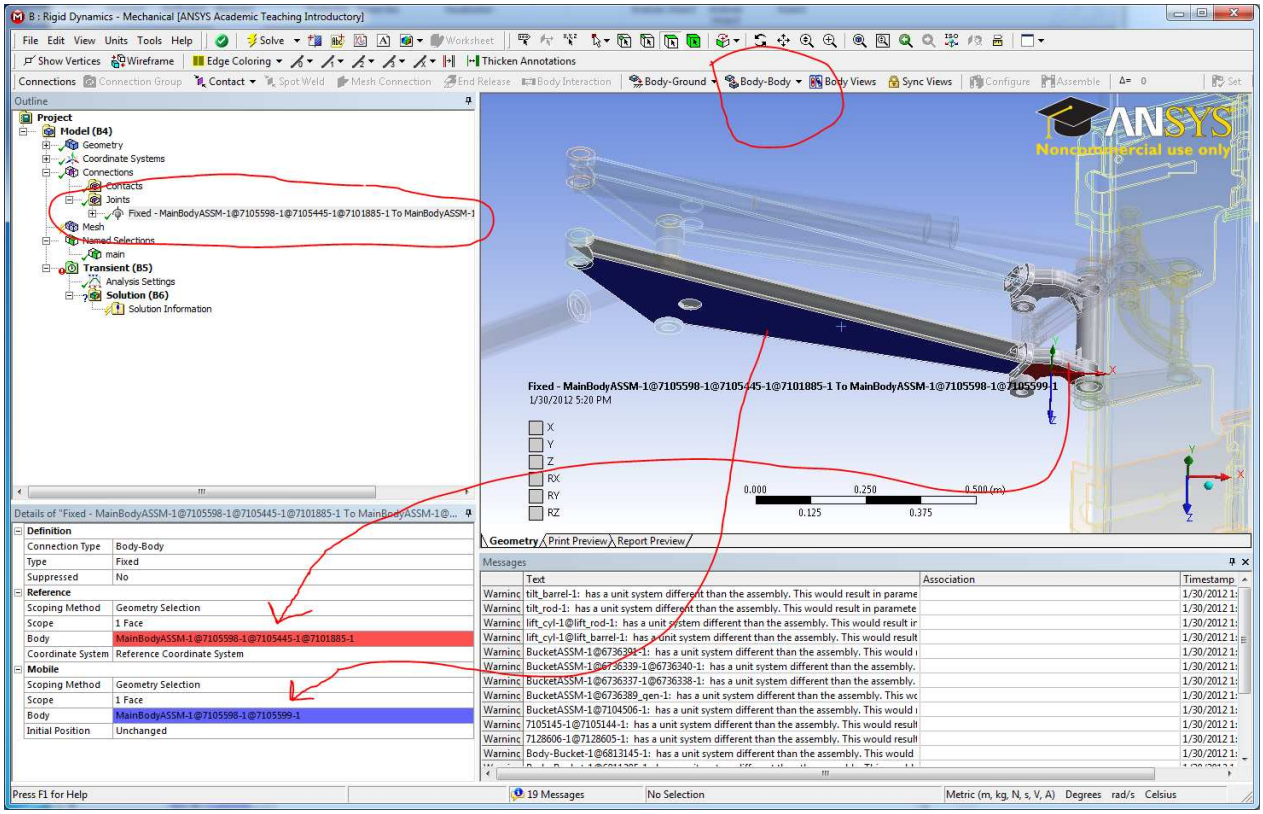
5. Once Ansys Mechanical is open, expand the Geometry tree and explore the different bodies in the model. Rename the parts to something meaningful. Some of these bodies need to be joined into one overall part and a good naming scheme is useful.



6. Expand the automatic contacts, select all the contacts and delete all the contacts connections in the tree by selecting delete from the right mouse button menu.



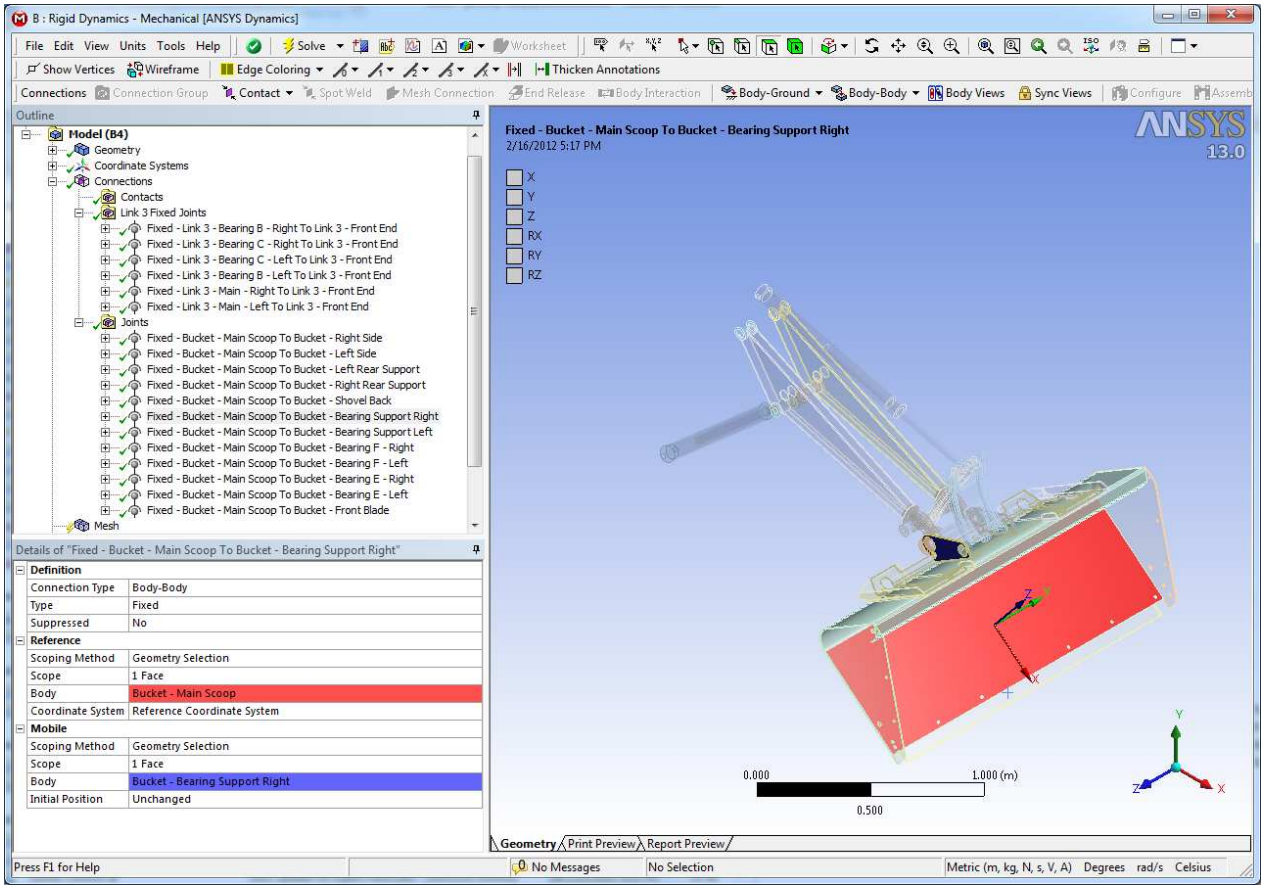
7. Select the Body-Body Connection menu and create fixed joints for all the parts that are rigid. For example, the MainAssembly has four bearing housings, two triangular pieces and a front piece. Use a fixed joint to connect these together. Yellow boxes need your attention and should be clicked on to proceed.



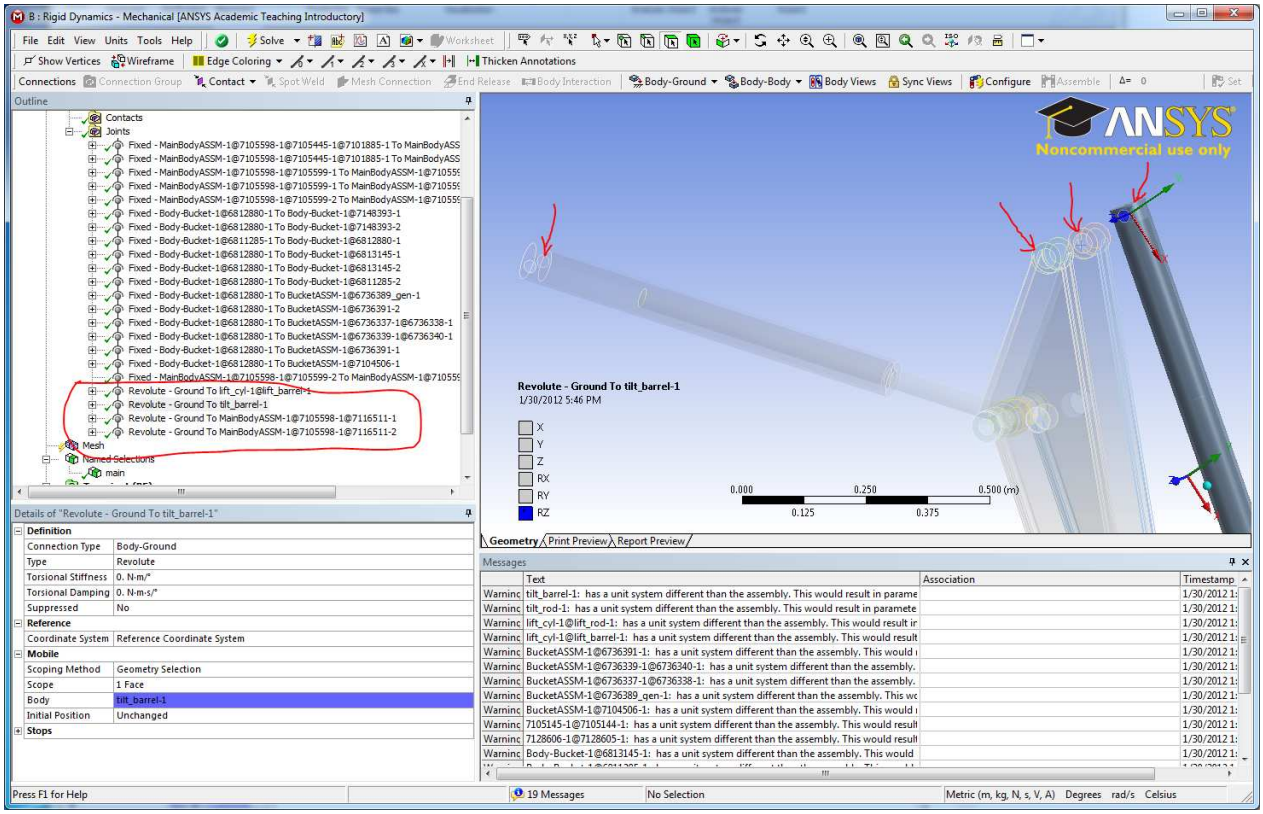
8. Since there are 7 parts for the MainBodyASSM, there needs to be 6 fixed joints to tie all the parts together. You can rename the folder containing these joints to something meaningful.

9. Right click on Connections and Insert a new Connection Group. Use this new group to make the bucket assembly.

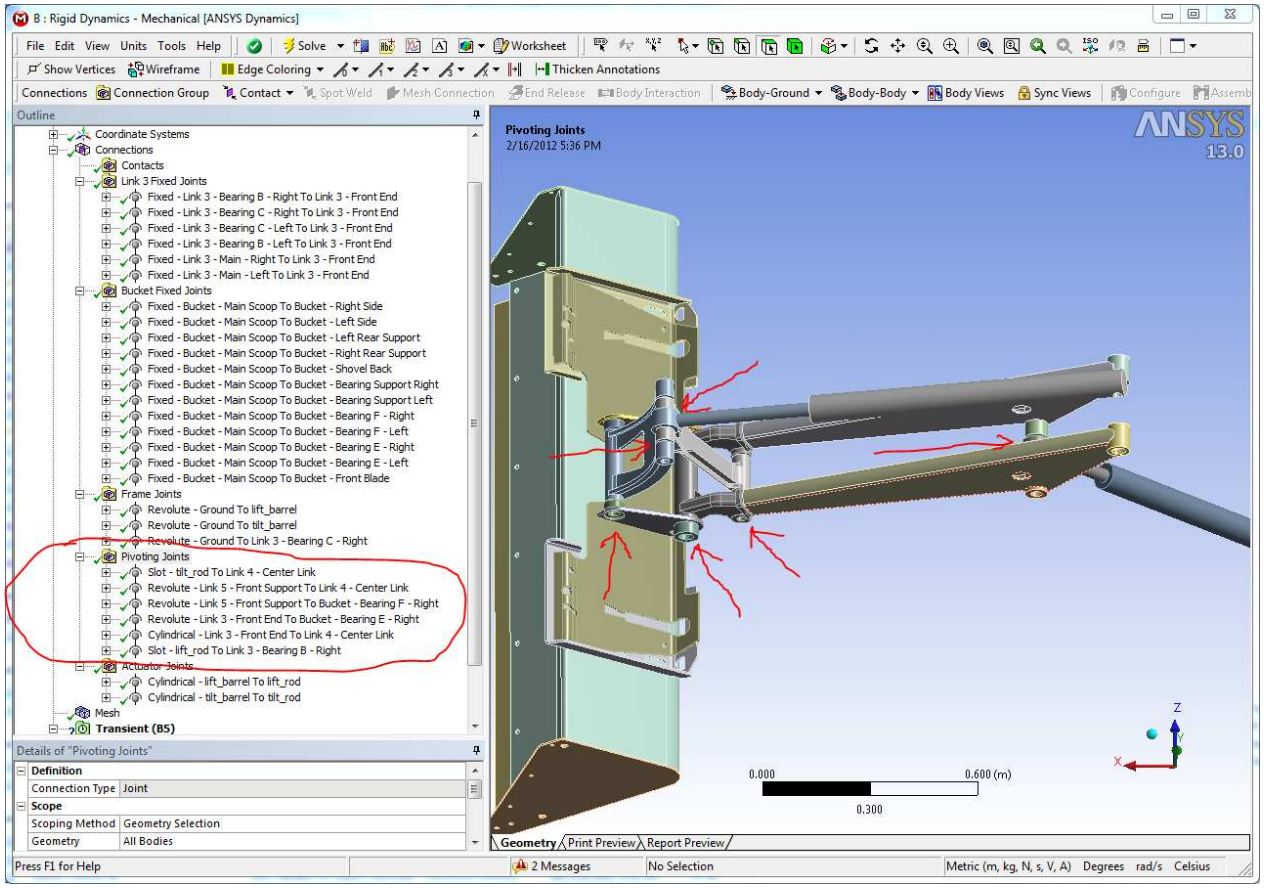
10. There are 13 parts to the bucket assembly, so 12 fixed joints are needed to tie the bucket parts together. Below shows this progress.


B : Rigid Dynamics - Mechanical [ANSYS Dynamics]
File Edit View Units Tools Help
Solve
Worksheet
Show Vertices
Wireframe
Edge Coloring
Thicken Annotations
Connections
Connection Group
Contact
Spot Weld
Mesh Connection
End Release
Body Interaction
Body-Ground
Body-Body
Body Views
Sync Views
Configure
Outline
Model (B4)
Geometry
Coordinate Systems
Connections
Contacts
Link 3 Fixed Joints
Fixed - Link 3 - Bearing B - Right To Link 3 - Front End
Fixed - Link 3 - Bearing C - Right To Link 3 - Front End
Fixed - Link 3 - Bearing C - Left To Link 3 - Front End
Fixed - Link 3 - Bearing B - Left To Link 3 - Front End
Fixed - Link 3 - Main - Right To Link 3 - Front End
Fixed - Link 3 - Main - Left To Link 3 - Front End
Joints
Fixed - Bucket - Main Scoop To Bucket - Right Side
Fixed - Bucket - Main Scoop To Bucket - Left Side
Fixed - Bucket - Main Scoop To Bucket - Left Rear Support
Fixed - Bucket - Main Scoop To Bucket - Right Rear Support
Fixed - Bucket - Main Scoop To Bucket - Shovel Back
Fixed - Bucket - Main Scoop To Bucket - Bearing Support Right
Fixed - Bucket - Main Scoop To Bucket - Bearing Support Left
Fixed - Bucket - Main Scoop To Bucket - Bearing F - Right
Fixed - Bucket - Main Scoop To Bucket - Bearing F - Left
Fixed - Bucket - Main Scoop To Bucket - Bearing E - Right
Fixed - Bucket - Main Scoop To Bucket - Bearing E - Left
Fixed - Bucket - Main Scoop To Bucket - Front Blade
Mesh
Details of "Fixed - Bucket - Main Scoop To Bucket - Bearing Support Right"
Definition
Connection Type Body-Body
Type Fixed
Suppressed No
Reference
Scoping Method Geometry Selection
Scope 1 Face
Body Bucket - Main Scoop
Coordinate System Reference Coordinate System
Mobile
Scoping Method Geometry Selection
Scope 1 Face
Body Bucket - Bearing Support Right
Initial Position Unchanged
Fixed - Bucket - Main Scoop To Bucket - Bearing Support Right
2/16/2012 5:17 PM
X
Y
Z
RX
RY
RZ
ANSYS
13.0
0.000
1.000 (m)
0.500
Geometry
Print Preview
Report Preview
Press F1 for Help
No Messages
No Selection
Metric (m, kg, N, s, V, A) Degrees rad/s Celsius

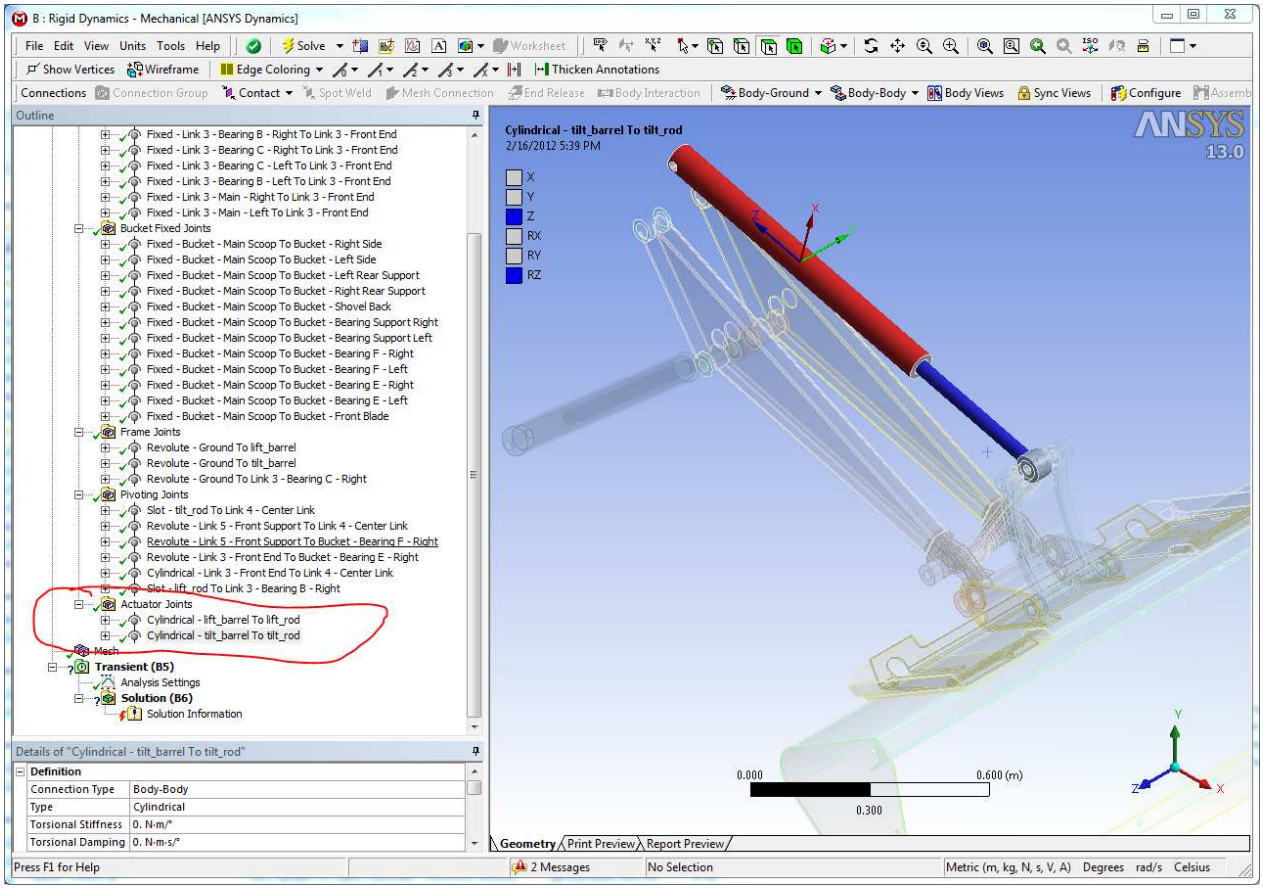
11. Create a new Joint folder and Add 3 Revolute Body-Ground Joints where the assembly attaches to the frame.



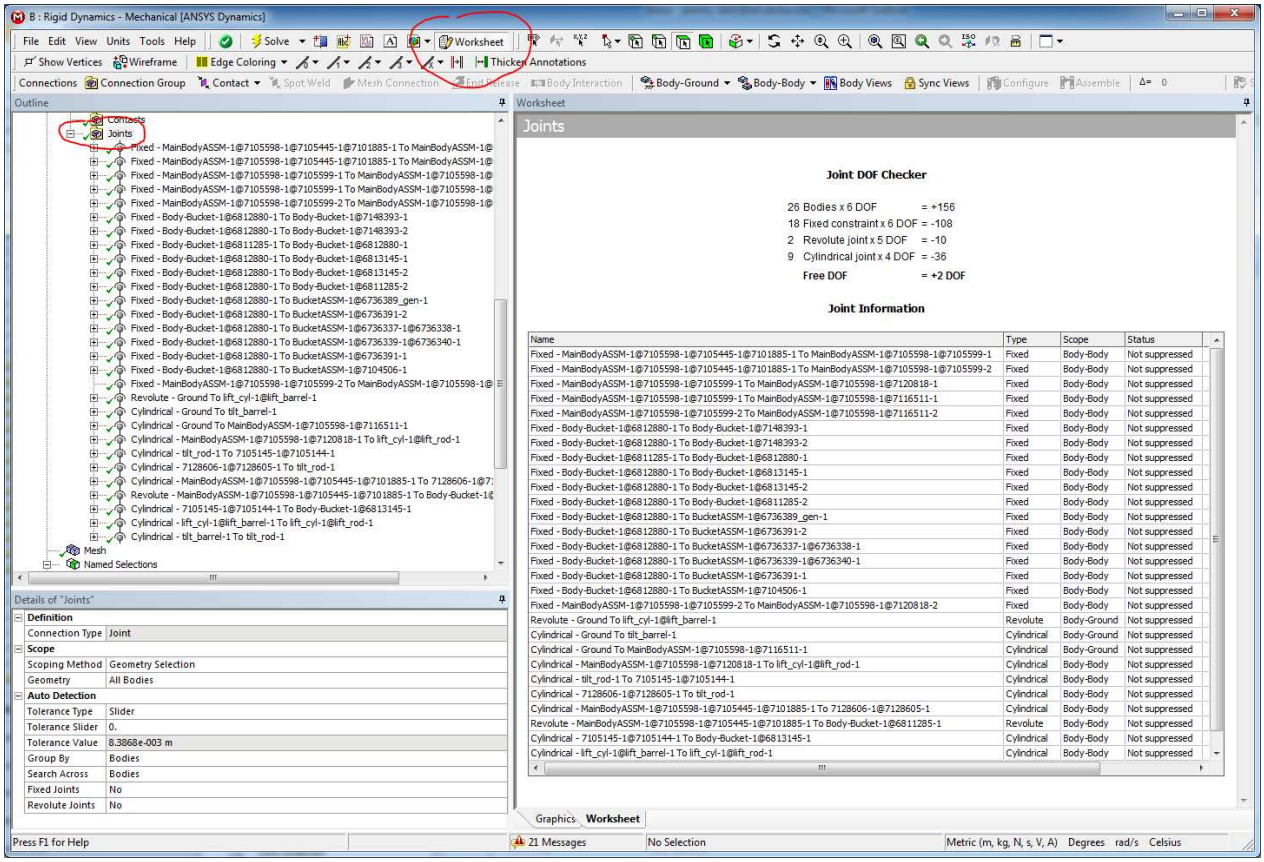
12. Add 6 Body-Body joints to represent the joints of the system.


B : Rigid Dynamics - Mechanical [ANSYS Dynamics]
File Edit View Units Tools Help
Solve
Worksheet
Show Vertices
Wireframe
Edge Coloring
Thicken Annotations
Connections
Connection Group
Contact
Spot Weld
Mesh Connection
End Release
Body Interaction
Body-Ground
Body-Body
Body Views
Sync Views
Configure
Outline
Coordinate Systems
Connections
Contacts
Link 3 Fixed Joints
Fixed - Link 3 - Bearing B - Right To Link 3 - Front End
Fixed - Link 3 - Bearing C - Right To Link 3 - Front End
Fixed - Link 3 - Bearing C - Left To Link 3 - Front End
Fixed - Link 3 - Bearing B - Left To Link 3 - Front End
Fixed - Link 3 - Main - Right To Link 3 - Front End
Fixed - Link 3 - Main - Left To Link 3 - Front End
Bucket Fixed Joints
Fixed - Bucket - Main Scoop To Bucket - Right Side
Fixed - Bucket - Main Scoop To Bucket - Left Side
Fixed - Bucket - Main Scoop To Bucket - Left Rear Support
Fixed - Bucket - Main Scoop To Bucket - Right Rear Support
Fixed - Bucket - Main Scoop To Bucket - Shovel Back
Fixed - Bucket - Main Scoop To Bucket - Bearing Support Right
Fixed - Bucket - Main Scoop To Bucket - Bearing Support Left
Fixed - Bucket - Main Scoop To Bucket - Bearing F - Right
Fixed - Bucket - Main Scoop To Bucket - Bearing F - Left
Fixed - Bucket - Main Scoop To Bucket - Bearing E - Right
Fixed - Bucket - Main Scoop To Bucket - Bearing E - Left
Fixed - Bucket - Main Scoop To Bucket - Front Blade
Frame Joints
Revolute - Ground To lift_barrel
Revolute - Ground To tilt_barrel
Revolute - Ground To Link 3 - Bearing C - Right
Pivoting Joints
Slot - tilt_rod To Link 4 - Center Link
Revolute - Link 5 - Front Support To Link 4 - Center Link
Revolute - Link 5 - Front Support To Bucket - Bearing F - Right
Revolute - Link 3 - Front End To Bucket - Bearing E - Right
Cylindrical - Link 3 - Front End To Link 4 - Center Link
Slot - lift_rod To Link 3 - Bearing B - Right
Actuator Joints
Cylindrical - lift_barrel To lift_rod
Cylindrical - tilt_barrel To tilt_rod
Mesh
Transient (B5)
Details of "Pivoting Joints"
Definition
Connection Type Joint
Scope
Scoping Method Geometry Selection
Geometry All Bodies
Pivoting Joints
2/16/2012 5:36 PM
ANSYS
13.0
0.000
0.600 (m)
0.300
Geometry
Print Preview
Report Preview
Press F1 for Help
2 Messages
No Selection
Metric (m, kg, N, s, V, A) Degrees rad/s Celsius

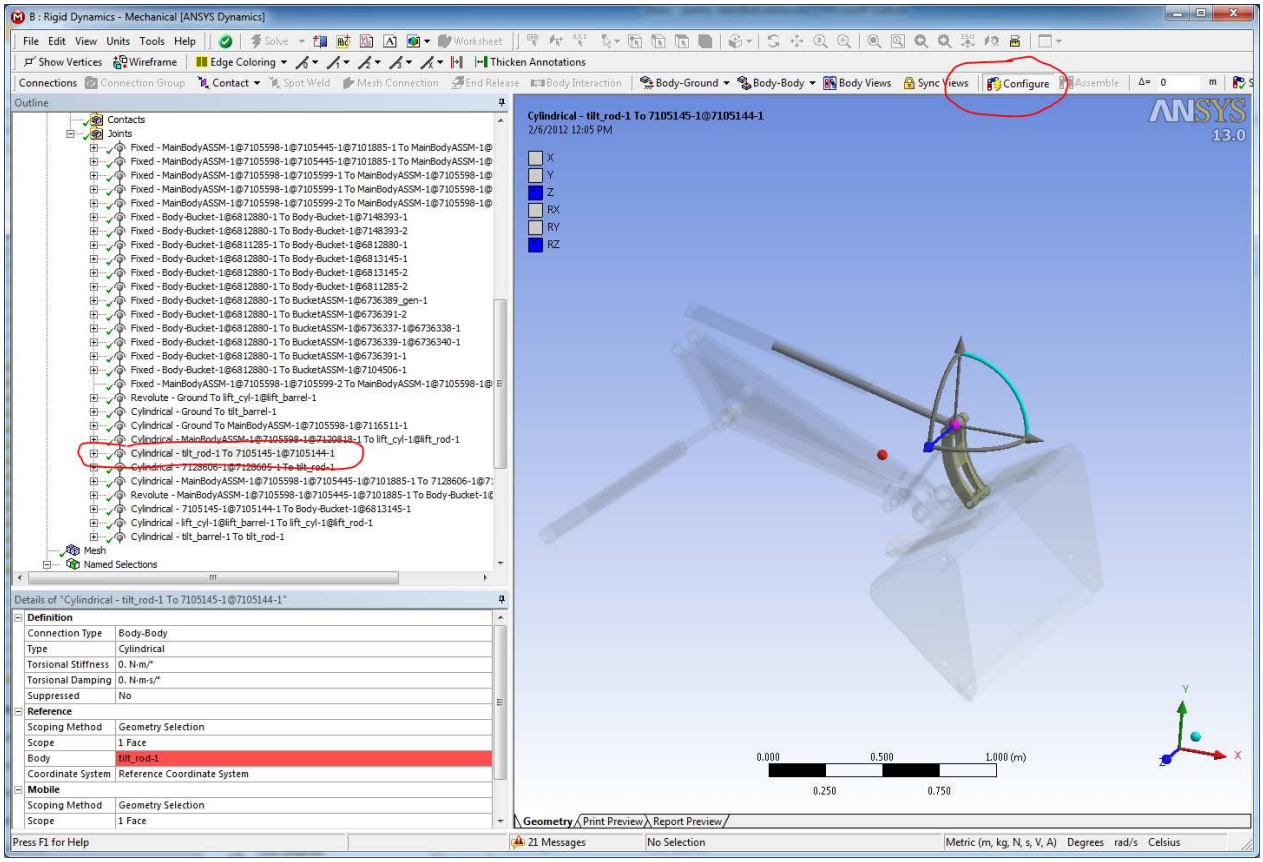
13. Add 2 Cylindrical joints to represent the actuator movement.


B : Rigid Dynamics - Mechanical [ANSYS Dynamics]
File Edit View Units Tools Help
Solve
Worksheet
Show Vertices
Wireframe
Edge Coloring
Thicken Annotations
Connections
Connection Group
Contact
Spot Weld
Mesh Connection
End Release
Body Interaction
Body-Ground
Body-Body
Body Views
Sync Views
Configure
Outline
Fixed - Link 3 - Bearing B - Right To Link 3 - Front End
Fixed - Link 3 - Bearing C - Right To Link 3 - Front End
Fixed - Link 3 - Bearing C - Left To Link 3 - Front End
Fixed - Link 3 - Bearing B - Left To Link 3 - Front End
Fixed - Link 3 - Main - Right To Link 3 - Front End
Fixed - Link 3 - Main - Left To Link 3 - Front End
Bucket Fixed Joints
Fixed - Bucket - Main Scoop To Bucket - Right Side
Fixed - Bucket - Main Scoop To Bucket - Left Side
Fixed - Bucket - Main Scoop To Bucket - Left Rear Support
Fixed - Bucket - Main Scoop To Bucket - Right Rear Support
Fixed - Bucket - Main Scoop To Bucket - Shovel Back
Fixed - Bucket - Main Scoop To Bucket - Bearing Support Right
Fixed - Bucket - Main Scoop To Bucket - Bearing Support Left
Fixed - Bucket - Main Scoop To Bucket - Bearing F - Right
Fixed - Bucket - Main Scoop To Bucket - Bearing F - Left
Fixed - Bucket - Main Scoop To Bucket - Bearing E - Right
Fixed - Bucket - Main Scoop To Bucket - Bearing E - Left
Fixed - Bucket - Main Scoop To Bucket - Front Blade
Frame Joints
Revolute - Ground To lift_barrel
Revolute - Ground To tilt_barrel
Revolute - Ground To Link 3 - Bearing C - Right
Pivoting Joints
Slot - tilt_rod To Link 4 - Center Link
Revolute - Link 5 - Front Support To Link 4 - Center Link
Revolute - Link 5 - Front Support To Bucket - Bearing F - Right
Revolute - Link 3 - Front End To Bucket - Bearing E - Right
Cylindrical - Link 3 - Front End To Link 4 - Center Link
Slot - lift_rod To Link 3 - Bearing B - Right
Actuator Joints
Cylindrical - lift_barrel To lift_rod
Cylindrical - tilt_barrel To tilt_rod
Mesh
Transient (B5)
Analysis Settings
Solution (B6)
Solution Information
Details of "Cylindrical - tilt_barrel To tilt_rod"
Definition
Connection Type Body-Body
Type Cylindrical
Torsional Stiffness 0. N-m/°
Torsional Damping 0. N-m-s/°
Cylindrical - tilt_barrel To tilt_rod
2/16/2012 5:39 PM
X
Y
Z
RX
RY
RZ
ANSYS
13.0
0.000
0.600 (m)
0.300
Geometry
Print Preview
Report Preview
Press F1 for Help
2 Messages
No Selection
Metric (m, kg, N, s, V, A) Degrees rad/s Celsius

14. When you think you are done creating all the joints, check the degrees of freedom (DOF) using the Joint DOF Checker. This mechanism should have 2 degrees of freedom (lift and tilt). You may have to change some of the joints to ensure compliance.

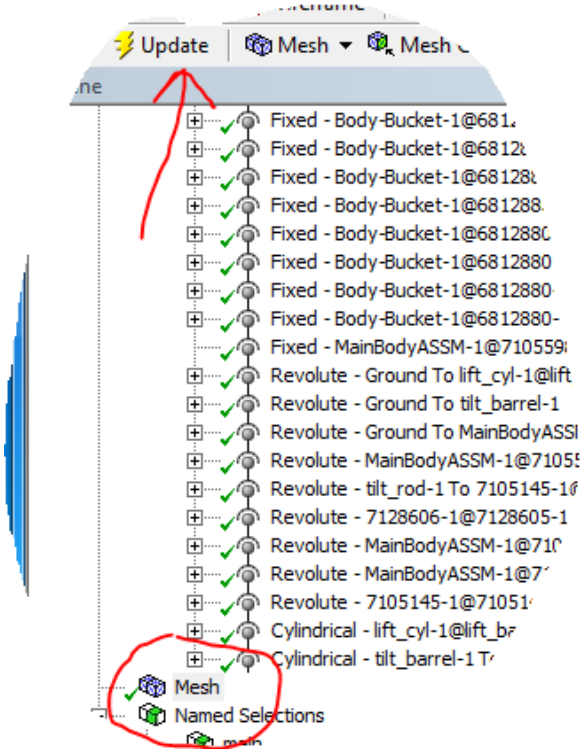

B : Rigid Dynamics - Mechanical [ANSYS Dynamics]
Worksheet
Joints
Joint DOF Checker
26 Bodies x 6 DOF = +156
18 Fixed constraint x 6 DOF = -108
2 Revolute joint x 5 DOF = -10
9 Cylindrical joint x 4 DOF = -36
Free DOF = +2 DOF
Joint Information

15. You can also click on a joint to configure it and test the joint defintion. This does not change the analysis configuration.

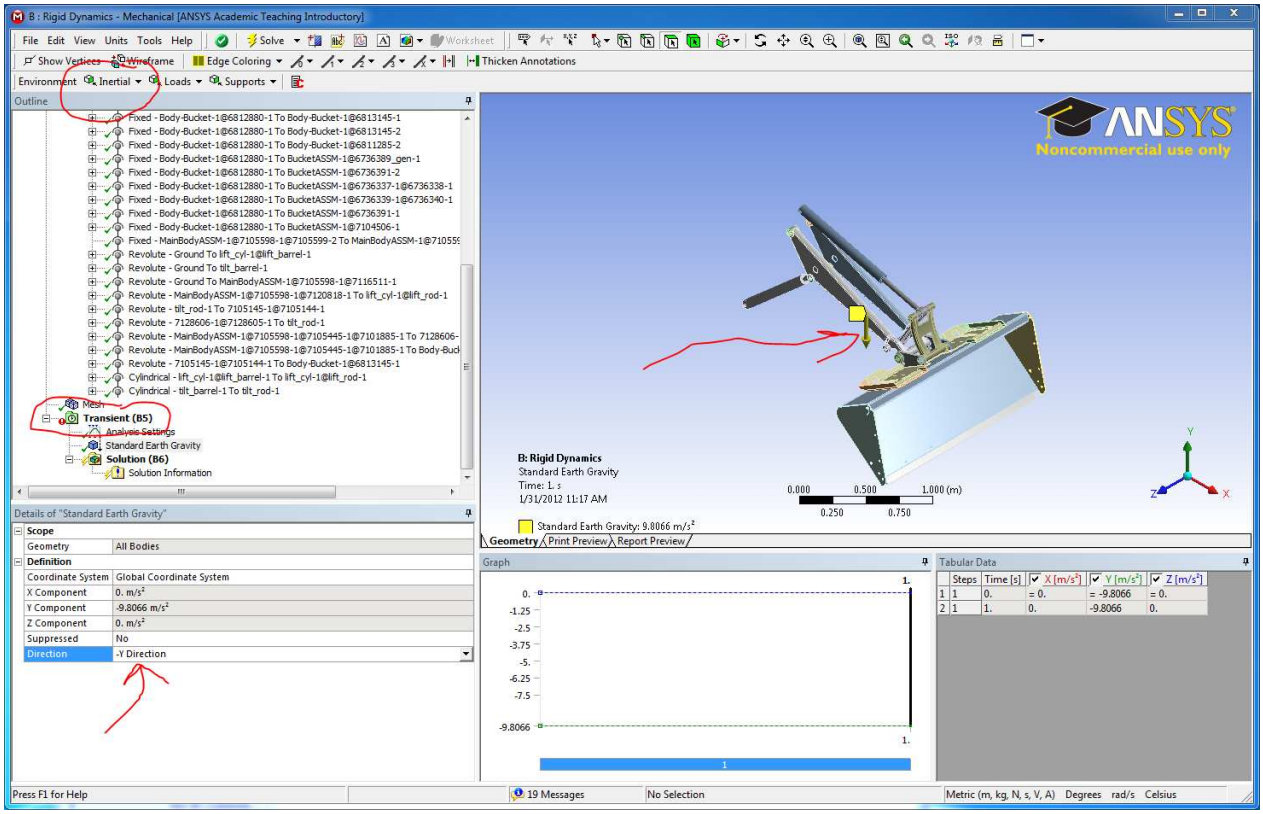


16. Save your project.

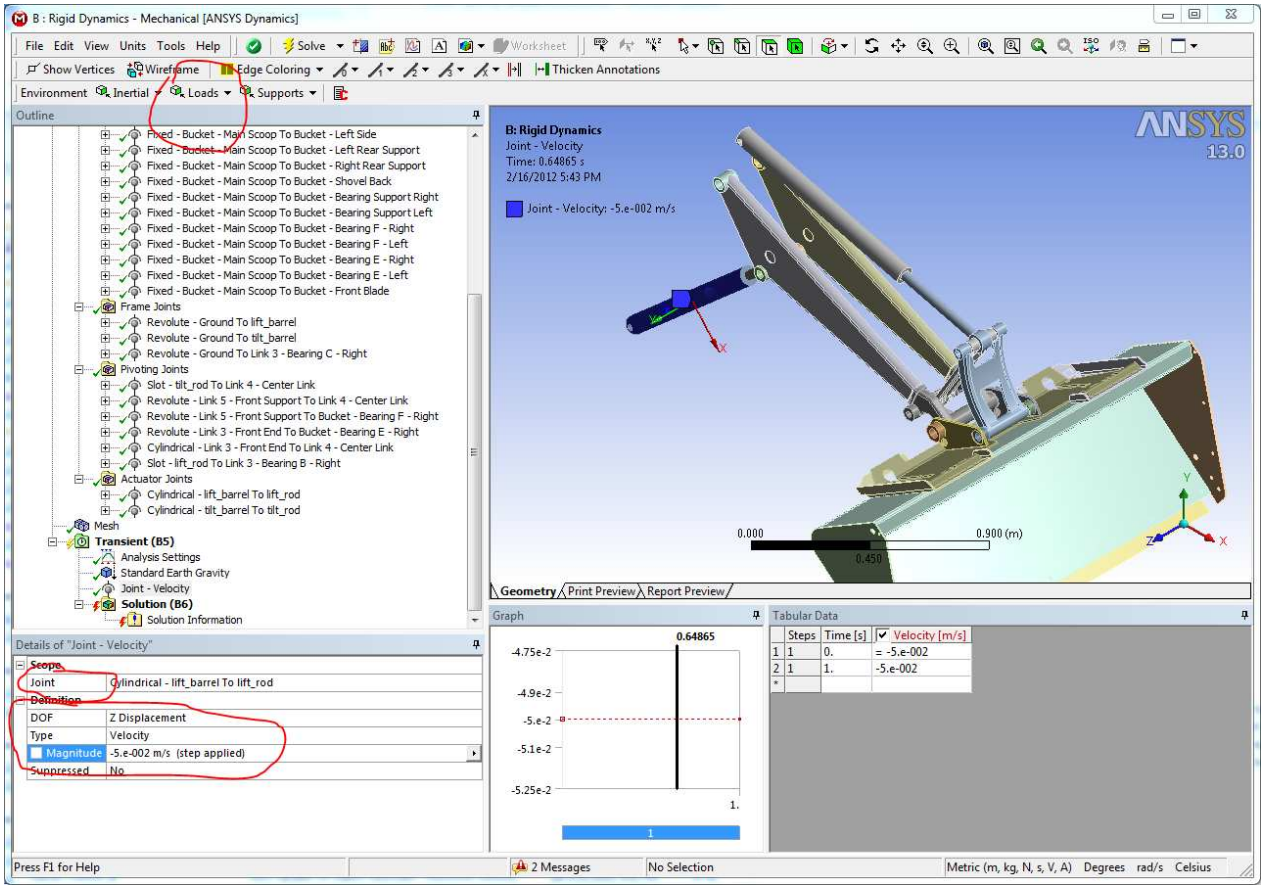
17. Mesh the project by clicking on the Mesh item in the tree and then Update.



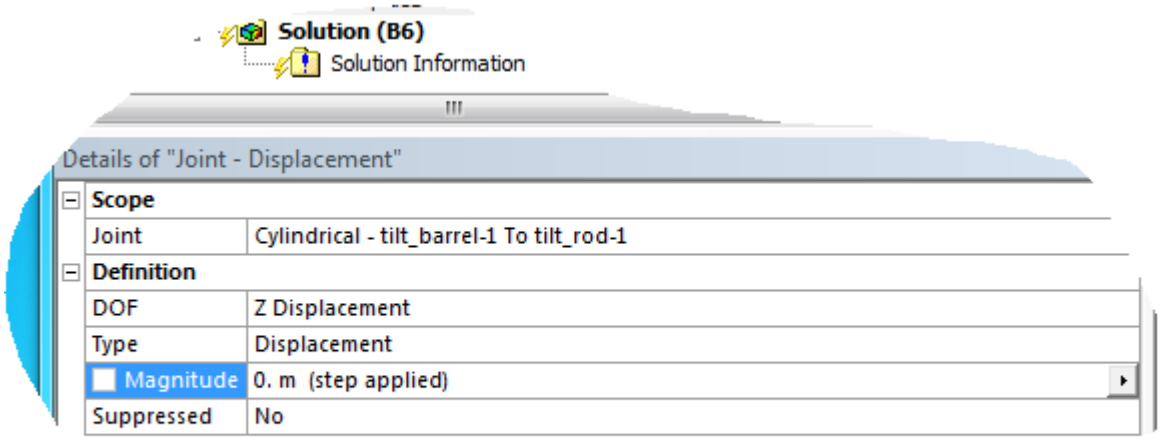
18. Click on Transient. Then click on Inertial and Select Standard Earth Gravity. Change the direction to -Y.



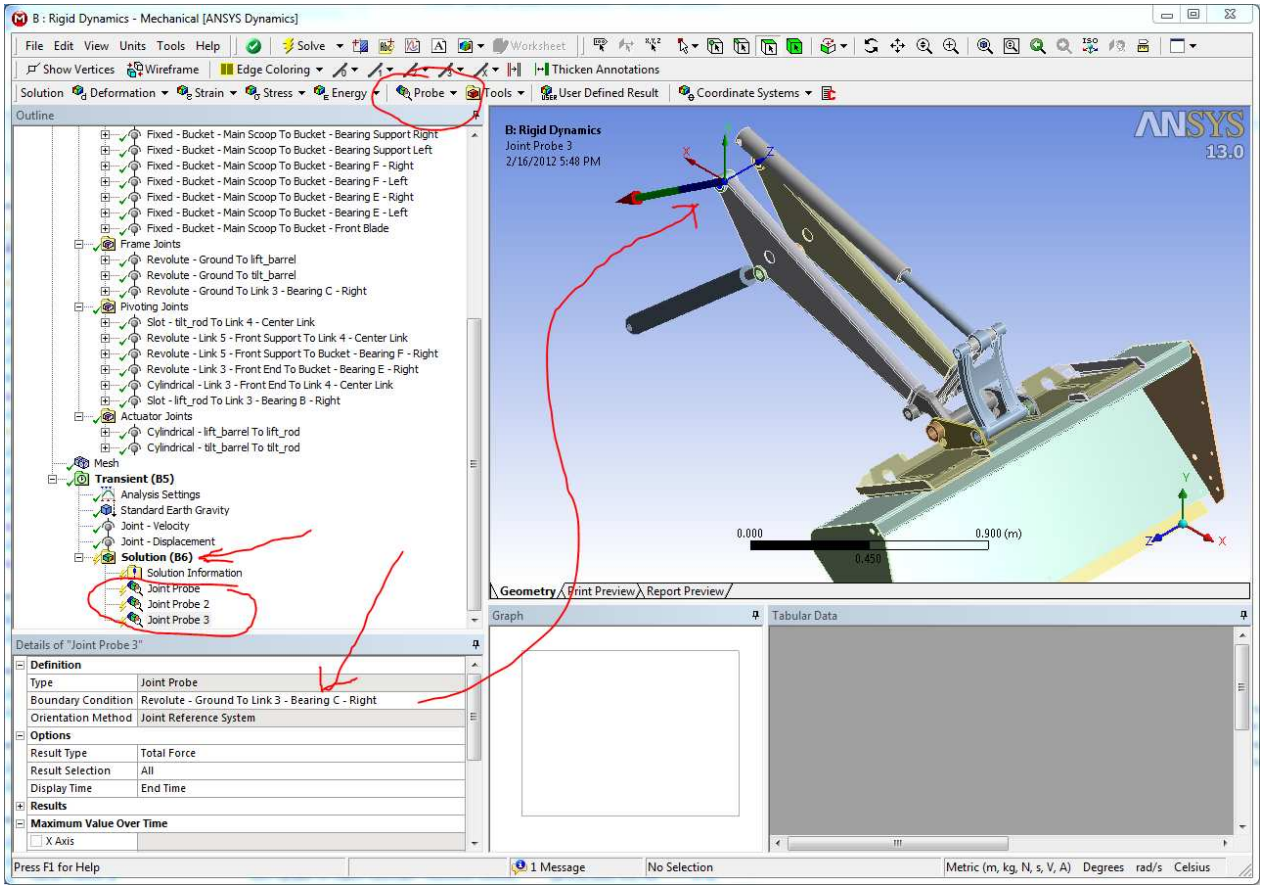
19. With Transient Selected, click on Loads and select Joint Load. Select the Cylindrical joint on the lift cylinder. Select the Z Displacement DOF. Select the Velocity Type, and apply a magnitude of -0.05.


B : Rigid Dynamics - Mechanical [ANSYS Dynamics]
File Edit View Units Tools Help
Solve
Show Vertices
Wireframe
Edge Coloring
Thicken Annotations
Environment
Inertial
Loads
Supports
Outline
Fixed - Bucket - Main Scoop To Bucket - Left Side
Fixed - Bucket - Main Scoop To Bucket - Left Rear Support
Fixed - Bucket - Main Scoop To Bucket - Right Rear Support
Fixed - Bucket - Main Scoop To Bucket - Shovel Back
Fixed - Bucket - Main Scoop To Bucket - Bearing Support Right
Fixed - Bucket - Main Scoop To Bucket - Bearing Support Left
Fixed - Bucket - Main Scoop To Bucket - Bearing F - Right
Fixed - Bucket - Main Scoop To Bucket - Bearing F - Left
Fixed - Bucket - Main Scoop To Bucket - Bearing E - Right
Fixed - Bucket - Main Scoop To Bucket - Bearing E - Left
Fixed - Bucket - Main Scoop To Bucket - Front Blade
Frame Joints
Revolute - Ground To lift_barrel
Revolute - Ground To tilt_barrel
Revolute - Ground To Link 3 - Bearing C - Right
Pivoting Joints
Slot - tilt_rod To Link 4 - Center Link
Revolute - Link 5 - Front Support To Link 4 - Center Link
Revolute - Link 5 - Front Support To Bucket - Bearing F - Right
Revolute - Link 3 - Front End To Bucket - Bearing E - Right
Cylindrical - Link 3 - Front End To Link 4 - Center Link
Slot - lift_rod To Link 3 - Bearing B - Right
Actuator Joints
Cylindrical - lift_barrel To lift_rod
Cylindrical - tilt_barrel To tilt_rod
Mesh
Transient (B5)
Analysis Settings
Standard Earth Gravity
Joint - Velocity
Solution (B6)
Solution Information
B: Rigid Dynamics
Joint - Velocity
Time: 0.64865 s
2/16/2012 5:43 PM
Joint - Velocity: -5.e-002 m/s
ANSYS
13.0
0.000
0.900 (m)
0.450
Geometry
Print Preview
Report Preview
Details of "Joint - Velocity"
Scope
Joint
Cylindrical - lift_barrel To lift_rod
Definition
DOF
Z Displacement
Type
Velocity
Magnitude
-5.e-002 m/s (step applied)
Suppressed
No
Graph
0.64865
-4.75e-2
-4.9e-2
-5.e-2
-5.1e-2
-5.25e-2
1.
Tabular Data
Steps
Time [s]
Velocity [m/s]
1 1 0. = -5.e-002
2 1 1. -5.e-002
Press F1 for Help
2 Messages
No Selection
Metric (m, kg, N, s, V, A) Degrees rad/s Celsius

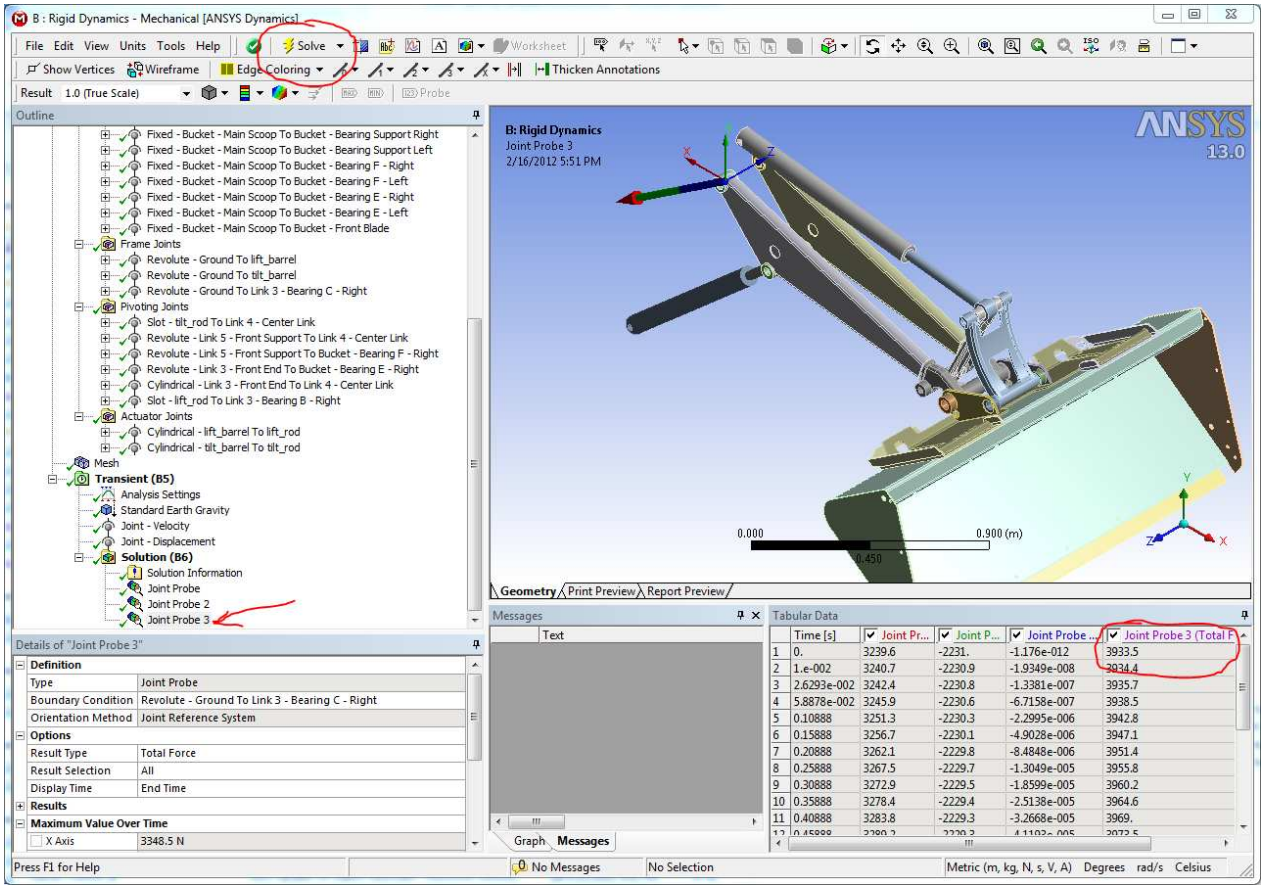
20. Apply a zero displacement to the tilt cylinder.



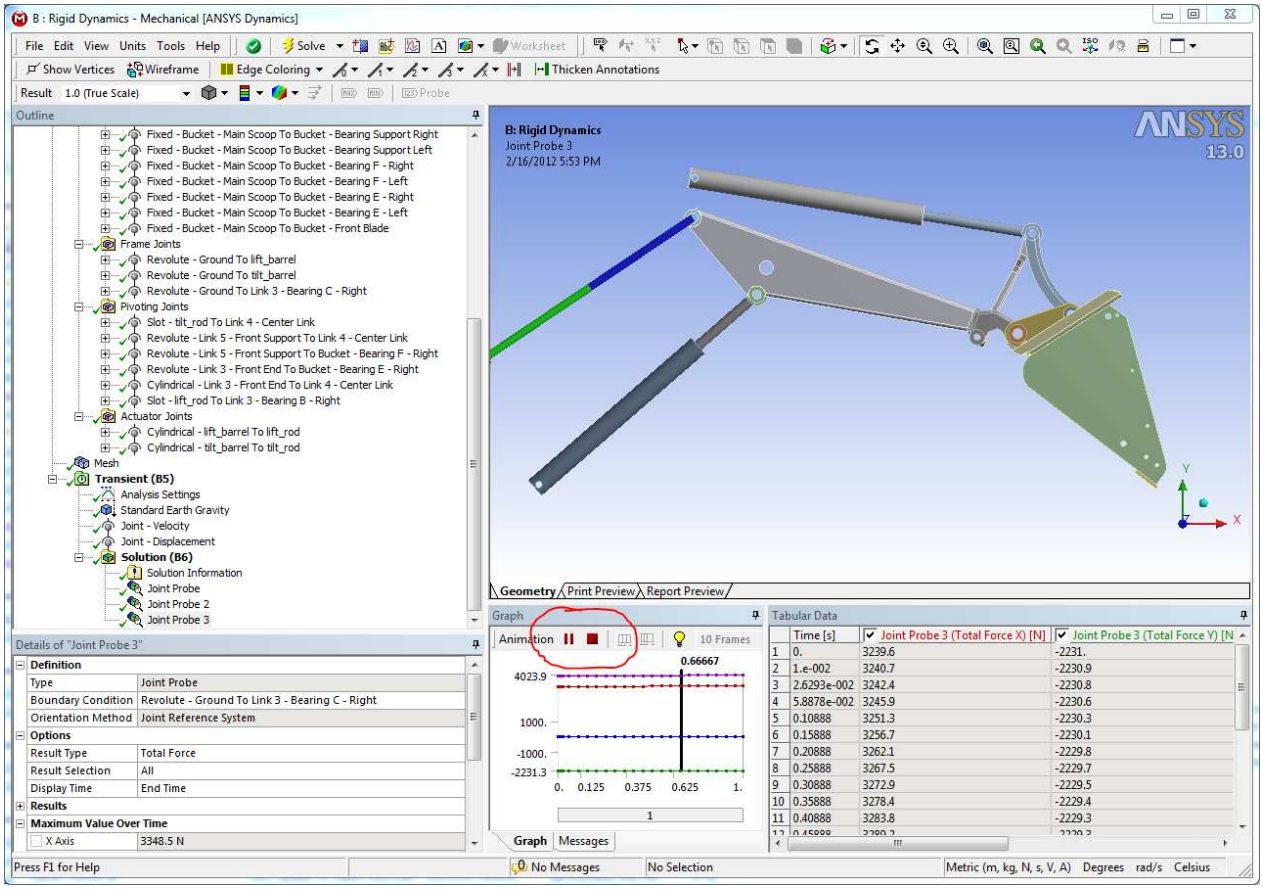
21. Create Solution output variables by selecting Solution from the tree. Select Joint Probes and probe the forces in each joint attached to the frame (Ground).


B : Rigid Dynamics - Mechanical [ANSYS Dynamics]
File Edit View Units Tools Help
Solve
Show Vertices
Wireframe
Edge Coloring
Thicken Annotations
Solution
Deformation
Strain
Stress
Energy
Probe
Tools
User Defined Result
Coordinate Systems
Outline
Fixed - Bucket - Main Scoop To Bucket - Bearing Support Right
Fixed - Bucket - Main Scoop To Bucket - Bearing Support Left
Fixed - Bucket - Main Scoop To Bucket - Bearing F - Right
Fixed - Bucket - Main Scoop To Bucket - Bearing F - Left
Fixed - Bucket - Main Scoop To Bucket - Bearing E - Right
Fixed - Bucket - Main Scoop To Bucket - Bearing E - Left
Fixed - Bucket - Main Scoop To Bucket - Front Blade
Frame Joints
Revolute - Ground To lift_barrel
Revolute - Ground To tilt_barrel
Revolute - Ground To Link 3 - Bearing C - Right
Pivoting Joints
Slot - tilt_rod To Link 4 - Center Link
Revolute - Link 5 - Front Support To Link 4 - Center Link
Revolute - Link 5 - Front Support To Bucket - Bearing F - Right
Revolute - Link 3 - Front End To Bucket - Bearing E - Right
Cylindrical - Link 3 - Front End To Link 4 - Center Link
Slot - lift_rod To Link 3 - Bearing B - Right
Actuator Joints
Cylindrical - lift_barrel To lift_rod
Cylindrical - tilt_barrel To tilt_rod
Mesh
Transient (B5)
Analysis Settings
Standard Earth Gravity
Joint - Velocity
Joint - Displacement
Solution (B6)
Solution Information
Joint Probe
Joint Probe 2
Joint Probe 3
B: Rigid Dynamics
Joint Probe 3
2/16/2012 5:48 PM
ANSYS
13.0
0.000
0.900 (m)
0.450
Geometry
Print Preview
Report Preview
Graph
Tabular Data
Details of "Joint Probe 3"
Definition
Type
Joint Probe
Boundary Condition
Revolute - Ground To Link 3 - Bearing C - Right
Orientation Method
Joint Reference System
Options
Result Type
Total Force
Result Selection
All
Display Time
End Time
Results
Maximum Value Over Time
X Axis
Press F1 for Help
1 Message
No Selection
Metric (m, kg, N, s, V, A) Degrees rad/s Celsius

22. Click the solve button then select a solution for a joint probe. Look for the total force in the first line of the last column.


B : Rigid Dynamics - Mechanical [ANSYS Dynamics]
File Edit View Units Tools Help
Solve
Worksheet
Show Vertices
Wireframe
Edge Coloring
Thicken Annotations
Result 1.0 (True Scale)
Probe
Outline
Fixed - Bucket - Main Scoop To Bucket - Bearing Support Right
Fixed - Bucket - Main Scoop To Bucket - Bearing Support Left
Fixed - Bucket - Main Scoop To Bucket - Bearing F - Right
Fixed - Bucket - Main Scoop To Bucket - Bearing F - Left
Fixed - Bucket - Main Scoop To Bucket - Bearing E - Right
Fixed - Bucket - Main Scoop To Bucket - Bearing E - Left
Fixed - Bucket - Main Scoop To Bucket - Front Blade
Frame Joints
Revolute - Ground To lift_barrel
Revolute - Ground To tilt_barrel
Revolute - Ground To Link 3 - Bearing C - Right
Pivoting Joints
Slot - tilt_rod To Link 4 - Center Link
Revolute - Link 5 - Front Support To Link 4 - Center Link
Revolute - Link 5 - Front Support To Bucket - Bearing F - Right
Revolute - Link 3 - Front End To Bucket - Bearing E - Right
Cylindrical - Link 3 - Front End To Link 4 - Center Link
Slot - lift_rod To Link 3 - Bearing B - Right
Actuator Joints
Cylindrical - lift_barrel To lift_rod
Cylindrical - tilt_barrel To tilt_rod
Mesh
Transient (B5)
Analysis Settings
Standard Earth Gravity
Joint - Velocity
Joint - Displacement
Solution (B6)
Solution Information
Joint Probe
Joint Probe 2
Joint Probe 3
B: Rigid Dynamics
Joint Probe 3
2/16/2012 5:51 PM
ANSYS
13.0
0.000
0.900 (m)
0.450
Geometry
Print Preview
Report Preview
Details of "Joint Probe 3"
Definition
Type
Joint Probe
Boundary Condition
Revolute - Ground To Link 3 - Bearing C - Right
Orientation Method
Joint Reference System
Options
Result Type
Total Force
Result Selection
All
Display Time
End Time
Results
Maximum Value Over Time
X Axis
3348.5 N
Messages
Text
Graph
Messages
Tabular Data
Time [s]
Joint Pr...
Joint P...
Joint Probe ...
Joint Probe 3 (Total F...
1 0. 3239.6 -2231. -1.176e-012 3933.5
2 1.e-002 3240.7 -2230.9 -1.9349e-008 3934.4
3 2.6293e-002 3242.4 -2230.8 -1.3381e-007 3935.7
4 5.8878e-002 3245.9 -2230.6 -6.7158e-007 3938.5
5 0.10888 3251.3 -2230.3 -2.2995e-006 3942.8
6 0.15888 3256.7 -2230.1 -4.9028e-006 3947.1
7 0.20888 3262.1 -2229.8 -8.4848e-006 3951.4
8 0.25888 3267.5 -2229.7 -1.3049e-005 3955.8
9 0.30888 3272.9 -2229.5 -1.8599e-005 3960.2
10 0.35888 3278.4 -2229.4 -2.5138e-005 3964.6
11 0.40888 3283.8 -2229.3 -3.2668e-005 3969.
Press F1 for Help
No Messages
No Selection
Metric (m, kg, N, s, V, A) Degrees rad/s Celsius

23. Click on the graph tab and press the play button for an animation.


B : Rigid Dynamics - Mechanical [ANSYS Dynamics]
File Edit View Units Tools Help
Solve
Show Vertices
Wireframe
Edge Coloring
Thicken Annotations
Result 1.0 (True Scale)
Outline
Fixed - Bucket - Main Scoop To Bucket - Bearing Support Right
Fixed - Bucket - Main Scoop To Bucket - Bearing Support Left
Fixed - Bucket - Main Scoop To Bucket - Bearing F - Right
Fixed - Bucket - Main Scoop To Bucket - Bearing F - Left
Fixed - Bucket - Main Scoop To Bucket - Bearing E - Right
Fixed - Bucket - Main Scoop To Bucket - Bearing E - Left
Fixed - Bucket - Main Scoop To Bucket - Front Blade
Frame Joints
Revolute - Ground To lift_barrel
Revolute - Ground To tilt_barrel
Revolute - Ground To Link 3 - Bearing C - Right
Pivoting Joints
Slot - tilt_rod To Link 4 - Center Link
Revolute - Link 5 - Front Support To Link 4 - Center Link
Revolute - Link 5 - Front Support To Bucket - Bearing F - Right
Revolute - Link 3 - Front End To Bucket - Bearing E - Right
Cylindrical - Link 3 - Front End To Link 4 - Center Link
Slot - lift_rod To Link 3 - Bearing B - Right
Actuator Joints
Cylindrical - lift_barrel To lift_rod
Cylindrical - tilt_barrel To tilt_rod
Mesh
Transient (B5)
Analysis Settings
Standard Earth Gravity
Joint - Velocity
Joint - Displacement
Solution (B6)
Solution Information
Joint Probe
Joint Probe 2
Joint Probe 3
B: Rigid Dynamics
Joint Probe 3
2/16/2012 5:53 PM
ANSYS
13.0
Geometry
Print Preview
Report Preview
Details of "Joint Probe 3"
Definition
Type Joint Probe
Boundary Condition Revolute - Ground To Link 3 - Bearing C - Right
Orientation Method Joint Reference System
Options
Result Type Total Force
Result Selection All
Display Time End Time
Results
Maximum Value Over Time
X Axis 3348.5 N
Graph
Animation
10 Frames
0.66667
4023.9
1000.
-1000.
-2231.3
0. 0.125 0.375 0.625 1.
Graph
Messages
Tabular Data
Time [s]
Joint Probe 3 (Total Force X) [N]
Joint Probe 3 (Total Force Y) [N]
1 0. 3239.6 -2231.
2 1.e-002 3240.7 -2230.9
3 2.6293e-002 3242.4 -2230.8
4 5.8878e-002 3245.9 -2230.6
5 0.10888 3251.3 -2230.3
6 0.15888 3256.7 -2230.1
7 0.20888 3262.1 -2229.8
8 0.25888 3267.5 -2229.7
9 0.30888 3272.9 -2229.5
10 0.35888 3278.4 -2229.4
11 0.40888 3283.8 -2229.3
Press F1 for Help
No Messages
No Selection
Metric (m, kg, N, s, V, A) Degrees rad/s Celsius

24. Compare the joint forces from ANSYS to the hand calculations for the locations that attach to the frame. Fill out the following table. What are the units for each variable?

	Hand Calculations			Ansys Results		
Frame Attachments	F_x	F_y	Total Force	F_x	F_y	Total Force
Lift Cylinder						
Main Arm						
Tilt Cylinder						

25. Turn in your hand calculations, the above table, and a screenshot of your Ansys model similar to the image in Step 23.

25 Rocket Lab Practical

25.1 Objective

A competent engineer should be able to satisfactorily estimate the performance of systems using at least three methods:

1. Analytical– develop equations needed to solve engineering problems
2. Numerical– determine numerical values that describe mechanical system properties and behavior
3. Experimental– measure and record system properties and behavior

This laboratory practical required sufficient competence in each field to complete the task. The analytical portion requires you to draw a free body diagram of the pendulum rocket assembly and determine the formula to calculate the acceleration based on mass and force. The numerical aspect requires implementation of a numerical integration to determine angular velocity from angular acceleration. The experimental verification requires you to measure a baseline performance and the performance of an altered pendulum. The pendulum will be altered by adding additional mass to the system.

The objective of the experiment is to determine the actual net thrust force applied to the air rocket, then to predict the maximum angular velocity of the system. The pendulum assembly will be augmented with known masses at various heights. Satisfactory completion of the lab requires the predicted maximum angular velocity to match the measured maximum angular velocity within 5% as calculated with the following formula:

$$\%\text{difference} = \frac{Actual - Predicted}{Actual} \times 100\%. \tag{25.1}$$

The predicted angle must be shown to the instructor or T.A. before the rocket is fired.

Every student is required to perform this laboratory individually within 3 hours. Preparation time is allowed outside the class room, but lab time extensions will not be given. Every student must

perform the exercise in their predetermined time slot. Completion of this laboratory exercise is required to pass the course.

25.2 Background and Theory

Will an air rocket actually propel a person as seen in the link below?

Human Rocket Video http://www.youtube.com/watch?v=oCaqAQYjtVw

To determine this, the thrust of the rocket must be measured. This section of the lab handout describes how to take acceleration and angle measurements to determine the time varying value of the net thrust. A rocket produces thrust by expanding gas through a nozzle. This expansion results in a velocity change of the gas. Since the gas has mass, this results in a change in momentum.

25.2.1 Thrust Calculation

The thrust of the rocket is dependent on pressure and pressure decreases as the air escapes through the nozzle. Therefore, the net thrust force depends on the pressure and if a force profile is to be reproduced, then the firing of the rocket should occur around the same pressure. The net thrust force also includes the effects of friction and misalignment in the nozzle. A free body diagram of the forces on the rocket is shown in Fig. 25.1. The point O is fixed with a bearing and an incremental optical encoder measures the angle θ. We can determine a value for F_{thrust} by summing the moments about point O.

$$\sum M_O = 0 = F_{thrust} L_F - w L_{cg} \sin\theta - I_O \ddot{\theta} \tag{25.2}$$

where the term $-I_o\ddot{\theta}$ is the inertial (D'Alembert) force from the acceleration of the rocket. The quantity I_o is the mass moment of inertia about the fixed pivot point and $\ddot{\theta}$ is the angular acceleration in radians per second per second. Since point O is assumed to be fixed, the angular acceleration is the quotient of linear acceleration in the transverse direction to the radius, or

$$\ddot{\theta} = \frac{a_\theta}{L_{accel}} \tag{25.3}$$

The accelerometer used to determine the acceleration is influenced by gravity, so the value of a_θ is measured by taking the reading from the x-direction of the accelerometer (See Fig. 25.2b) and

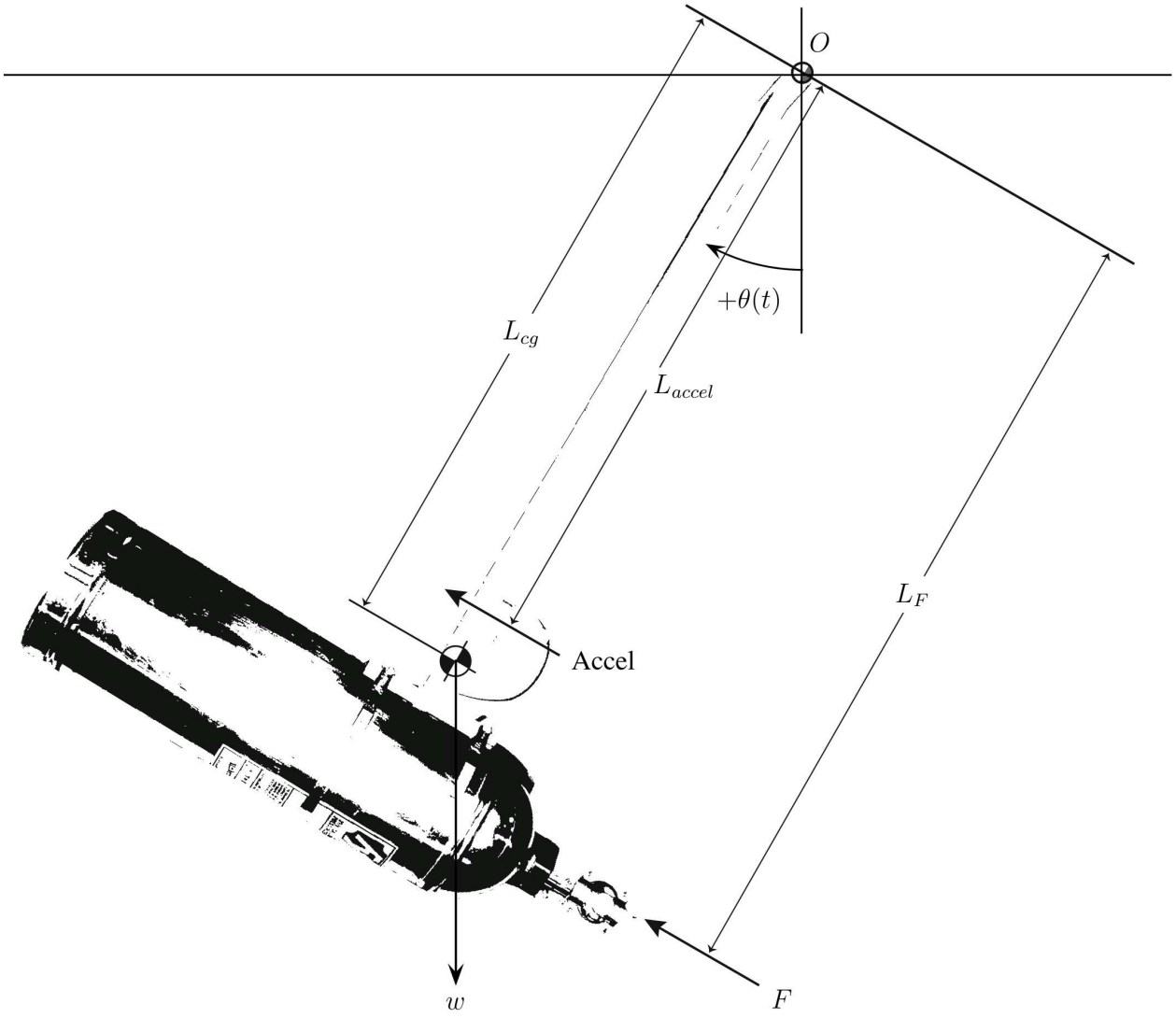
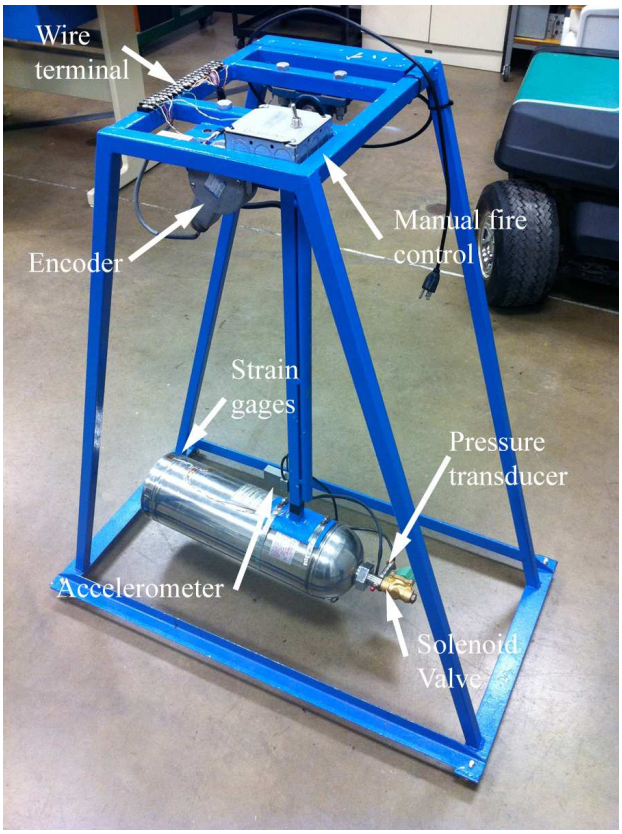
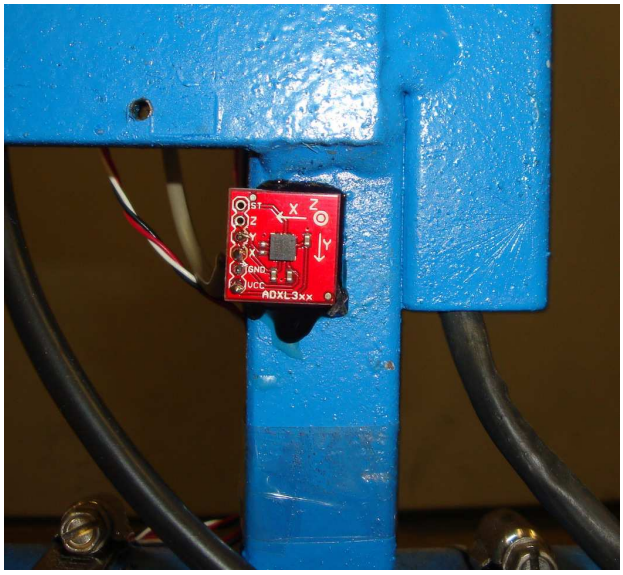


Figure 25.1: Free body diagram of the rocket for thrust calculation.



(a) Components of the rocket lab apparatus.



(b) Close-up photo of the orientation of the accelerometer.

Figure 25.2: Lab Apparatus.

Rocket	Weight, w (lb)	L_{cg} (in)	L_{accel} (in)	L_F (in)
A	15.15	30.3	32.0	38.7
B	13.53	31.2	32.0	38.8
C	13.58	31.1	32.0	38.9

Magnet	Weight (lb)
1	1.816
2	1.812
3	1.815
4	1.818

Table 25.1: Weight and length data for the different apparatus including wires and sensors. Each item has a letter or number to distinguish it from the others.

subtracting the component due to gravity:

$$\ddot{\theta} = \frac{a_x - g\sin\theta}{L_{accel}} \tag{25.4}$$

where a_x is the acceleration measured by the accelerometer and ϕ is the initial tilt angle of the accelerometer. Substituting Eq. (25.4) into Eq. (25.2) and solving for thrust force yields

$$F_{thrust} = \frac{I_O[a_x - g\sin\theta]}{L_F L_{accel}} + \frac{wL_{cg}\sin\theta}{L_F} \tag{25.5}$$

It is important to note that the values of F_{thrust}, a_x, and θ all depend on time. Furthermore, a_x and θ are measured with an accelerometer and encoder, respectively. The length quantities can be measured directly and the weight of the assembly was determined during assembly. Each apparatus is slightly different and the data for the lengths and weight are shown in Table 25.1.

It is important to realize that the force applied is dependent on the initial pressure. Therefore, to have consistent results, the rocket must be fired at the same pressure each time. This can be achieved by firing the rocket at the same voltage from the pressure sensor each time. The pressure sensor is connected and displayed in LabVIEW.

25.2.2 Mass Moment of Inertia

The resistance to rotation of the rigid pendulum about point O is quantified by the polar moment of inertia. If the pendulum is free to rotate with no applied force, the equation of motion from summing the moments about point O is

$$I_O\ddot{\theta} + wL_{cg}\sin\theta = 0 \tag{25.6}$$

where I_O is the mass moment of inertia, w is the total weight of the rocket assembly (from Table 25.1 on page 25-5), and L_{cg} is the distance from the point of rotation to the center of mass (from Table 25.1). If small angles are used, then $\sin\theta \approx \theta$ when measured in radians. Making this substitution gives a constant coefficient 2nd order differential equation:

$$I_O\ddot{\theta} + wL_{cg}\theta = 0 \tag{25.7}$$

which can be rearranged to the form:

$$\ddot{\theta} + \omega_n^2\theta = 0 \tag{25.8}$$

where $\omega_n^2 = \frac{wL_{cg}}{I_O}$ is the square of the natural frequency. For small damping, the observed frequency is similar to the natural frequency and can be determined by measuring the period of oscillation. The period relates to the natural frequency as

$$T = \frac{2\pi}{\omega_n} \tag{25.9}$$

Therefore, the mass moment of inertia can be determined as

$$I_O = \frac{wL_{cg}T^2}{4\pi^2} \tag{25.10}$$

where I_O is the polar moment of inertia,
w is the total weight of the rotating rocket assembly,
L_{cg} is the length from the pivot point to the center of mass, and
T is the period of oscillation.

Units of I_O should be equivalent to mass-length^2.

Part of this lab is to add extra masses and change the weight, moment of inertia, and center of gravity. If a lumped mass is added to the pendulum assembly, the new moment of inertia can be estimated as

$$I_{O,new} = I_{O,baseline} + mr^2 \tag{25.11}$$

This assumes the added masses have negligible moments of inertia about their centroids. If multiple masses are added, then Eq. 25.11 becomes

$$I_{O,new} = I_{O,baseline} + \sum_{i=1}^{n} m_i r_i^2 \tag{25.12}$$

where $I_{O,baseline}$ is determined with no masses using Eq. 25.10, m_i is the value of the ith mass, and r_i is the distance of the center of the ith mass from the center of rotation. The change in the center of mass location is left to the student as part of the lab exercise, if needed.

25.2.3 Angular Velocity Prediction

The objective of the lab is to predict the maximum angular velocity after modifying the inertial properties of the pendulum. The predicted maximum angular velocity must match the maximum angular velocity data recorded by the encoder to complete the lab.

The angular velocity can be determined numerically by integrating the angular acceleration. In other words, the angular velocity is determined as

$$\dot{\theta}(t) = \dot{\theta}_o + \int_0^t \ddot{\theta}(\tau)\, d\tau \tag{25.13}$$

where τ is an dummy variable for integration and $\dot{\theta}_o$ is the initial angular velocity. If the assembly starts from rest, then the angular acceleration is assumed to be constant, then

$$\dot{\theta}(t) = \dot{\theta_{t-1}} + \ddot{\theta}_{t-1}\Delta t$$

is zero. The trapezoidal rule is a good candidate for the integration as shown in the following formula:

$$\int_{t_0}^{t_n} f(t)\, dt \approx \sum_{i=0}^{n-1} \frac{f(t_i) + f(t_{i+1})}{2}\Delta t \tag{25.14}$$

where $f(t)$ is a an arbitrary function of time. The trapezoidal rule can be easily implemented in a spreadsheet or computer program.

The angular acceleration is determined from the torque applied to the pendulum. It is computed as the cross product of the thrust force and the perpendicular distance from the center of rotation

to the thrust vector. Since the force is determined from Eq. (25.5) is based on measurements (not analytical functions), the integration must be performed numerically.

25.3 Example Numerical Integration for a Simple Pendulum

To complete this lab, some numerical integration is required. This section provides an example of simple numerical integration

25.4 LabVIEW Data Acquisition

A LabVIEW program will be available to accomplish the tasks needed to obtain the measurements for this lab. As a starting point, the following VI should be considered an example when acquiring the data. The y-axis accelerometer is optional for this experiment. This VI can be obtained from the shared space at S:\ENS\Mechanical Engineering\Daily\Machine Dynamics Labs\

DAQ Assistant

There are two DAQ assistants used to obtain the analog and the encoder information. The pressure transducer uses a custom voltage with excitation from an NI 9237, the accelerometer uses a voltage input from the NI 9215, and the encoder uses a counter input from the NI 9411. All modules need to share the same time clock that exists on the NI9237 module. This is known as correlated timing.

Sample Compression

Acquires a large number of data points and compresses the data points into a smaller number of points.

This Express VI is configured as follows:

Reduction Factor: 100

Reduction Method: Mean

Collector

Collects input signals and returns the most recent data, up to the specified maximum number of samples per channel. When you call this Express VI repeatedly and the Express VI reaches the maximum number of samples per channel, the Express VI discards the oldest data and adds the newest data to the collected samples.

———————

This Express VI is configured as follows:

Number of Points to Collect: 1500

25.5 Instrument Hookup and Calibration

25.5.1 Accelerometer

Wire the x-axis of the accelerometer to the analog input of an analog input module in the CompactDAQ. The wiring diagram for the apparatus is shown in Table 25.3 on page 25-10.

The accelerometer is a dual axis MEMS base sensor produced by Analog Devices shown in Fig. 25.2b on page 25-4. The device is part number ADXL-322 and the datasheet is freely available on the Internet. The accelerometer will be calibrated using at least five points against gravity. The relationship between the voltage and the acceleration is as follows:

$$a = mV + b \tag{25.15}$$

where a is the acceleration felt by the sensor (including gravity), V is the voltage, m is the gain, and b is the offset.

When calibrating against gravity, the initial tilt of the accelerometer needs to be determined as well as the gain and offset. The quadrature encoder produces accurate angles, so the acceleration sensed is

$$a = -g\sin(\theta) \tag{25.16}$$

#	Color	Description
20	Red	Strain Remote Sense (+)
19	White	Strain Analog Out (+)
18	Black	Strain Excite (+)
17	Red	Strain Remote Sense (-)
16	White	Strain Analog Out (-)
15	Black	Strain Excite (-)
14	Yellow or Gr/Wt	Accel Output Y
13	Blue or Green	Accel Output X
12	Black or Blue	Accel Ground
11	Red or lt blue	Accel Vcc (+5V)
10	Lt.Brown, Br/Wt	Pressure Sensor Out(-)
9	Brown	Pressure Sensor Out(+)
8	White or Or/Wt	Pressure Sensor EX(-)
7	Orange	Pressure Sensor EX(+)
6	Solid Color Wire	Solenoid Valve Control
5	Green	Quadrature Channel Z
4	White	Quadrature Channel A
3	Blue	Quadrature Channel B
2	Red or Blue	+5V
1	White or Black	Common

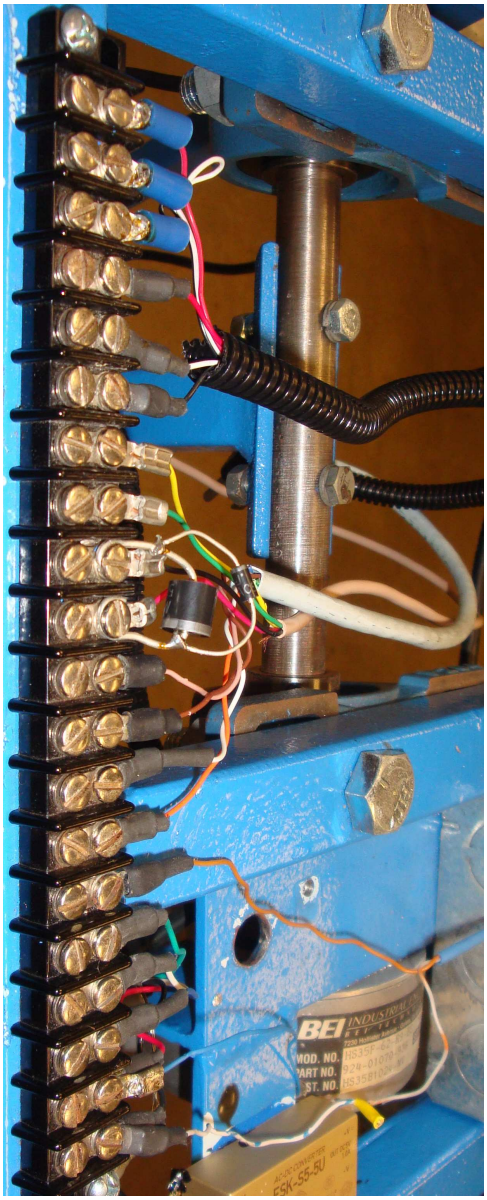


Figure 25.3: Wiring diagram for the Rocket Lab apparatus.

where g is the acceleration due to gravity, θ is the angle from the encoder and ϕ is the initial tilt angle. Combining Eqs. (25.15) with (25.16) gives

$$-g\sin(\theta) = mV + b \tag{25.17}$$

which can be used to calibrate the accelerometer. The angle is obtained from the encoder with zero being the initial at rest state. Subsequent angles are determined when the assembly is supported with a wooden bar in an angled position and not moving. For each angle, the raw voltage should be obtained from the accelerometer. An implementation in Excel is shown in Fig. 25.4 on page 25-12.

25.5.2 Pressure Transducer

The pressure transducer uses a diaphragm to force a plunger. The force measured using strain gages on the plunger is proportional to the pressure. Since the force us determined using strain, the pressure transducer is also based on the Wheatstone Bridge Circuit. Both the accelerometer and the pressure transducer require a bridge based signal conditioner. For this lab, the NI 9237 module provides the necessary signal conditioning for gathering data. The pressure sensor does not need to be calibrated since it is only used to help with repeatability.

25.5.3 LabVIEW Programming

Write a program that will fire the rocket and calculate the impulse from the data provided by the instruments. Demonstrate this project works with a live firing of the rocket.

25.5.4 Quadrature Encoder

A quadrature encoder works by looking at two digital signals that are 90 degrees out of phase. These signals are denoted channel A and channel B. If Channel A leads channel B, then the rotation is in one direction. If the direction is reversed, then channel B will leading channel A. Counting the number of pulses that transpire gives the angle of revolution. The issue of these encoders is it requires high speed counting circuits to work well and not miss pulses. Specialty integrated circuits (IC's) exist to perform the counting and reduce the computing requirements of the host DAQ system. For this lab, an NI 9411 in the Compact DAQ will be used.

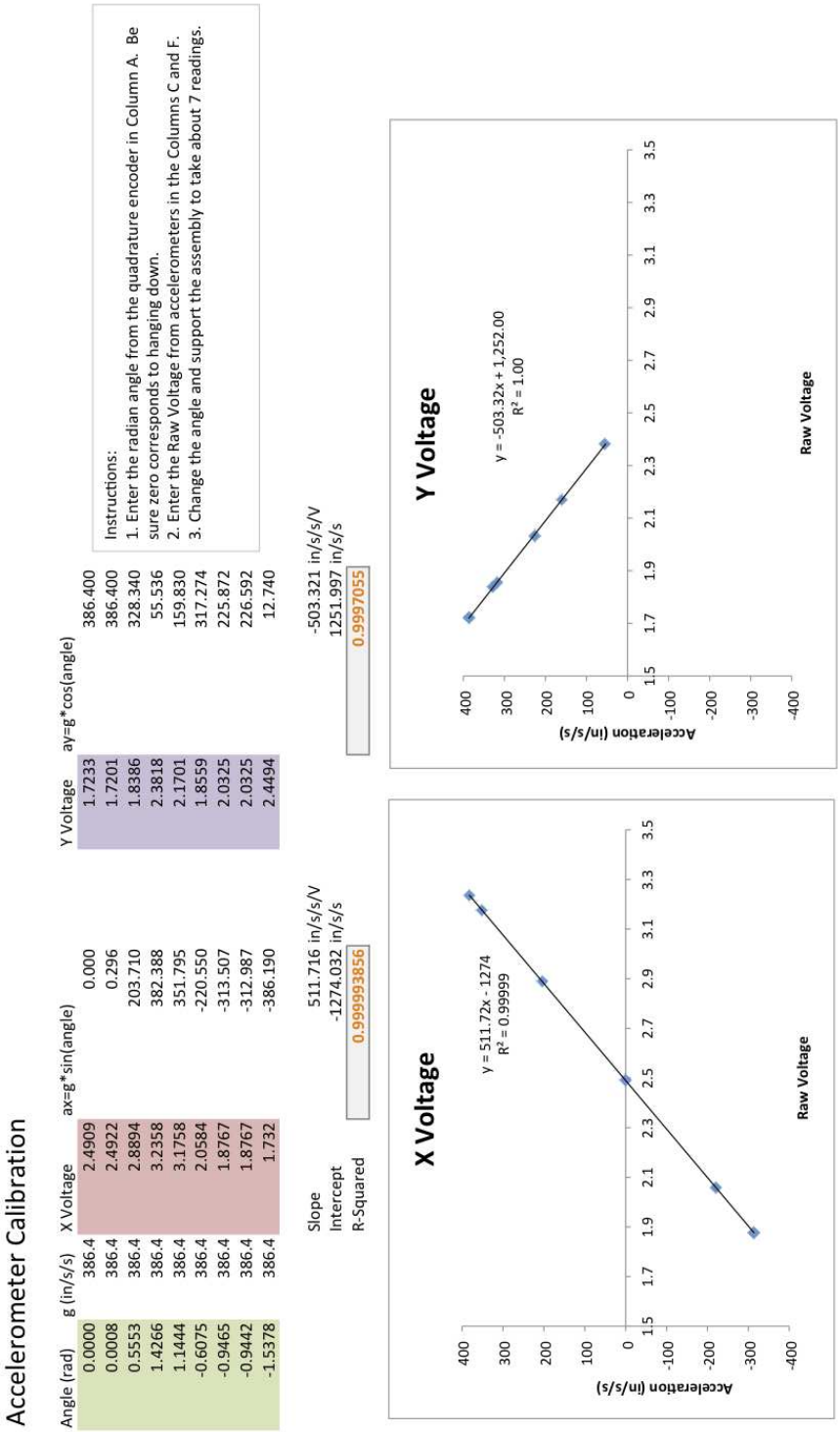

Accelerometer Calibration
Angle (rad)
g (in/s/s)
X Voltage
ax=g*sin(angle)
Y Voltage
ay=g*cos(angle)
0.0000 386.4 2.4909 0.000 1.7233 386.400
0.0008 386.4 2.4922 0.296 1.7201 386.400
0.5553 386.4 2.8894 203.710 1.8386 328.340
1.4266 386.4 3.2358 382.388 2.3818 55.536
1.1444 386.4 3.1758 351.795 2.1701 159.830
-0.6075 386.4 2.0584 -220.550 1.8559 317.274
-0.9465 386.4 1.8767 -313.507 2.0325 225.872
-0.9442 386.4 1.8767 -312.987 2.0325 226.592
-1.5378 386.4 1.732 -386.190 2.4494 12.740
Slope 511.716 in/s/s/V
Intercept -1274.032 in/s/s
R-Squared 0.999993856
Slope -503.321 in/s/s/V
Intercept 1251.997 in/s/s
R-Squared 0.9997055
Instructions:
1. Enter the radian angle from the quadrature encoder in Column A. Be sure zero corresponds to hanging down.
2. Enter the Raw Voltage from accelerometers in the Columns C and F.
3. Change the angle and support the assembly to take about 7 readings.
X Voltage
y = 511.72x - 1274
R² = 0.99999
Acceleration (in/s/s)
Raw Voltage
Y Voltage
y = -503.32x + 1,252.00
R² = 1.00
Acceleration (in/s/s)
Raw Voltage

Figure 25.4: A printout of an example calibration worksheet for the dual axis accelerometers using different angles from the encoder. The commands =SLOPE(), =INTERCEPT(), and =RSQ() were used to build this sheet.

25.6 Procedure

Please read all these instruction before starting the procedure.

1. Ensure the LabVIEW program is working and acuiring data. An instructor or TA can help with this.
2. Calibrate the accelerometer using different values of gravity as described in section 25.5.1. When the assembly is still, the acceleration in x should average around zero.
3. Type in a filename and some comments.
4. Rock the pendulum assembly about 20 degrees and record the free oscillation to determine the period. The period is output by the LabVIEW program. Use this information in Eq. (25.10) to determine I_O.
5. Fill the tank with air until pressure in the tank is the same as the supply. In other words, fill the tank until you do not hear the air moving.
6. Be sure all signals are being acquired and the rocket is not moving. If the pressure is dropping, be sure it is doing so at a readable rate.
7. When the pressure sensor voltage reaches a predetermined value, fire the rocket with the toggle switch. No remote firing is required.
8. Before the dynamic data leaves the collector buffers in the screen, stop the DAQ loop to save the file.
9. Open the comma separated value (CSV) file saved by LabVIEW in another program like Excel or Matlab.
10. Calculate the net thrust force and the net torque. This is the baseline thrust for a given initial pressure.
11. Predict the maximum angular velocity of the assembly if additional masses, m_i, are added to the pendulum rod at positions r_i. Both the center of gravity and the moment of inertia of the complete assembly will change.
12. Call an instructor or TA to apply magnets to your assembly and change the C.G. Measure the locations of the additional mass and determine the new inertial properties.
13. Calculate a new maximum angular velocity and a minimum angular velocity. Have the instructor or TA write it on the whiteboard.

14. Refill the rocket with air and repeat the data acquisition. Compare the new maximum angular velocities with the one predicted in the previous step. Determine the amount of difference using Eq. (25.1).
15. If the difference is less than 5%, then the reports and graphs can be prepared. If the difference is not within 5%, then the lab data aquisition or prediction must be performed again. Every student must complete this laboratory within 3 hours. Failure to complete the laboratory will result in an failing grade in the course.
16. Turn in a graph with three plots: 1) the base-line anglular velocity data, 2) the measured anglular velocity data with the additional mass, and 3) the predicted anglular velocity with the additional mass. Be sure the plot is labeled correctly and has your name on it. Also, indicate the relative difference between the maximum velocity predicted and the maximum velocity measured.

Part IV

Supplemental Material

A Grading Sheet for Technical Reports

All technical reports must follow the style guidelines for the Society of Automotive Engineers. The Word template file and further description can be obtained from the SAE website:

http://volunteers.sae.org/#authors

A style guide is also available from http://volunteers.sae.org/authors/styleguide.pdf and must be used. The following page has a grade sheet template for written reports.

Technical Report Grade Sheet

Name of Student:

Item	Available Points	Earned Points
Follow SAE Formatting Guidelines		
Text is organized according to guidelines	3	
Figures have numbers and titles	3	
Tables have numbers and titles	3	
Appendix has numbers and titles	2	
Section headings are formatted correctly	3	
Contains a susinct Abstract	4	
Has a conclusion section	4	
Equations are numbered	3	
Writing		
Applicable material in each section	8	
Neatness	7	
Organization of points	8	
Clarity of writing	7	
Interpretation		
Covered important points	15	
Correct interpetation	10	
Creativity	5	
Sample calculations	8	
Uncertainty analysis	7	
Penalties		
Exceeded page length	-10	
Insufficient references (must have at least 3)	-10	
Late (20% per day)	$-20\times$ _____ days	
Total	100	

B Participation Survey

After completing a technical report or group exercise, please fill out the following form. The participation rate must add to 100. For example, if there were four people in a group and they all did an equal amount for the project, then the participation rates would be 25% for everyone.

Item	Answer	Participation Rate
Project Title:		
My Name:		
Co-Author 1:		
Co-Author 2:		
Co-Author 3:		
Co-Author 4:		
Co-Author 5:		
Total		100%

C Formal Letter Template

Haywood G. Student
123 Memory Lane
Left Arm Pit, AR 87653

November 1, 2009

Jeremy S. Daily, Ph.D., P.E.
Assistant Professor of Mechanical Engineering
University of Tulsa
600 S. College Ave.
Tulsa, OK 74104

In regards to: <Put in an appropriate description here>

Dear Dr. Daily:

This is the body text of a formal letter describing the findings in a laboratory exercise. This letter is an extended abstract and must contain key elements. Those elements include an objective, procedure, results, discussion, and conclusion. Since space is limited, more thought is required and copying elements from the lab handout is wasted space. The mathematics should be limited and only reflect equations that are not in the lab handout.

A well presented graph of your results must be enclosed with this letter. Also, the raw data you obtain in lab should be attached. This demonstrates the actual process of gathering data. Tables, spreadsheets, and Matlab code are not to be submitted with this letter.

The formal letter must be formatted as shown here. It is not allowed to exceed 2 pages of 12 point font with 1" margins all the way around. Be sure to sign your letter when finished. In Microsoft Word, you can use the Letter Wizard found in the Tools > Letters and Mailings menu. The output of the wizard will most likely need additional formatting.

A formal letter is a very important tool of communicating. Our goal at the University of Tulsa is to give you the means for effective communication so you represent yourself well and reflect positively on the University of Tulsa.

Respectfully,

Haywood G. Student
Junior Mechanical Engineering Student
University of Tulsa

Enclosures (2)
1. Graph of ...
2. Handwritten data collection sheet

Bibliography

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Index

360 degree method, 12-18

absolute motion, 7-28
acceleration, 5-9, 7-2, 10-2
acceleration factor, 9-2, 9-6
accuracy, 5-19
ampere, 5-7
angle measurement, 5-11
angular displacement, 14-8
angular momentum, 14-21
angular velocity, 14-8
articulated vehicle, 8-1
atmosphere, 5-12

balances, 5-5
bumper gun, 9-14

candela, 5-8
center of mass, 10-16
centrifugal force, 14-4
centripetal acceleration, 14-4
closed system, 13-11
closing speed, 7-28
coefficient of friction, 9-1, 10-5
coefficient of restitution, 11-16, 14-37
collinear collision, 12-2
COLM, 11-9
conservation of energy, 13-10
conservation of linear momentum, 11-8, 11-9, 12-1
conservative force, 13-11
cross product, 6-13

deceleration, 7-2
density, 5-13
diagram, 10-6
direction cosine, 11-10
displacement, 7-1
distance, 7-1
dot product, 6-11, 13-6
drag factor, 9-2, 9-6
drag factors, 9-4
drag sled accuracy, 9-12
drag sleds, 9-10
drag tables, 9-10
dummy variable, 7-17
dynamic friction, 10-5
dynamic weight shift, 10-28

earth-fixed coordinates, 8-1
elastic, 11-15
electric current, 5-7
energy, 5-13, 11-4, 13-6

equilibrium, 10-10

factors of ten, 5-2
flat bottom sleds, 9-10
foot, 5-4
force, 5-9, 10-3
free body diagram, 10-6
friction, 9-1, 10-5
fundamental units, 5-3

global coordinates, 8-1
gyration, 14-14

heading, 12-2
Hooke's Law, 13-2

ideal spring, 13-2
impact circle, 12-17, 12-34
impulse, 11-2
in-line collision, 12-2
inelastic, 11-15
inertia, 10-1
inertial force, 10-26, 10-27
irreversible work, 13-9
isolated system, 13-11

jerk, 7-3
Joule, 5-14

kelvin, 5-7
kilogram, 5-4
kinematics, 7-1
kinetic energy, 13-6, 14-19
kinetic friction, 10-5

lateral acceleration, 14-3
length, 5-3
lock-up factor, 9-8

mass, 5-4, 10-2
mass moment of inertia, 14-9
measurement, 5-1
mechanical energy, 13-10
meter, 5-3
mole, 5-8
moment, 5-14, 10-7
moment of inertia, 14-9
momentum, 11-2, 11-4, 11-8, 12-1
multiple departures, 12-38

Newton, Sir Issac, 10-1
newton, unit of force, 5-10
non-conservative force, 13-12
numerical precision, 5-20

occupant kinematics, 12-21
orthogonal, 10-9

parallel axis theorem, 14-12
percentage of braking, 9-8
potential energy, 13-8
pound, 5-10
pound-mass, 5-5
poundal, 5-10
power, 5-15, 13-17
precision, 5-19
pressure, 5-12
Principal Direction of Force, 12-2, 12-18
projectile motion, 8-8
pseudo force, 10-26
Pythagorean Theorem, 6-5

radian, 5-11
radius of gyration, 14-14
reaction force, 10-4

rectilinear motion, 7-3
reference frame, 7-28
relative difference, 5-18
relative motion, 7-28
restitution, 11-16, 14-37
resultant, 12-12
resultant vector, 6-7
reversible work, 13-9
rotational motion, 14-7
round bottom sleds, 9-11
rounding, 5-22

scalar, 6-2
scalar product, 6-11
second, 5-5
secondary impacts, 12-39
sensitivity, 7-28
shot marker, 9-14
SI, 5-2
significant figures, 5-21
slugs, 5-4
speed, 7-1
springs, 13-2
static friction, 10-5
stress, 5-13
superelevation, 9-21

temperature, 5-7
terminal point, 6-3
time, 5-5
torque, 5-14, 6-15, 10-7, 14-14
truncating, 5-23

uncertainty, 5-24
uniform circular motion, 14-1
uniform projectile motion, 8-8
units, base, 5-3
units, derived, 5-8

vector, 6-2
vector components, 6-18
vector diagram, 12-11
vehicle-fixed coordinates, 8-1
velocity, 7-1

weight, 10-2
work, 5-13, 13-1
Work–Energy Theorem, 13-14